Primes

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1. What are the integers made up of?

   (a) If your only operation is plus, "+", what integers do you need to make all the integers?

   (b) If your only operation is multiplication, what integers do you need to make all the positive integers?

2. What if we restricted ourselves to the even integers? In other words, 6 would be in our set but 3 would not. This would make 6 a "prime" since it can not be written as a product of two even integers.

   (a) What are the "primes" if our number system is the even integers?

   (b) In the set of even integers, what is a prime factorization of the following integers? In particular, see if you can find different prime factorizations.

$$\begin{aligned} 4 &= \\ 8 &= \\ 12 &= \\ 16 &= \\ 20 &= \\ 24 &= \\ 28 &= \\ 32 &= \\ 36 &= \\ 40 &= \\ 60 &= \\ 420 &= \end{aligned}$$

   (c) What if we restricted to odd numbers? What would the "primes" be in this case?

   (d) (Back to the integers.) What does it mean for a number to have a unique prime factorization?

3. How many primes are there? There are lots of proofs that there are an inifinite number of primes many of which can be found here: `http://primes.utm.edu/notes/proofs/infinite/`.

   Here is one of the more popular proofs that there are an infinite number of primes (due to Euclid). Suppose there were only a finite number of primes. Then we could list them:

   $$2, 3, 5, 7, 11, 13, \ldots, P$$

   where P is the largest prime. Then, consider the number:

   $$N = 2 \cdot 3 \cdot 5 \cdot 7 \cdots P + 1$$

   Then you notice that $N > P$ and if you divide N by one of our primes we get a remainder of 1. In other words, none of our primes divides N and therefore N should have been on our list of primes

   (a) Consider the numbers (motivated by Euclid's proof) and determine if they are prime or composite (find the prime factorizations):

   $$2 + 1 = 3$$
   $$2 \cdot 3 + 1 = 7$$
   $$2 \cdot 3 \cdot 5 + 1 = 31$$
   $$2 \cdot 3 \cdot 5 \cdot 7 + 1 = 211$$
   $$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 + 1 = 2311$$
   $$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 + 1 = 30031$$
   $$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 + 1 = 510511$$
   $$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 + 1 = 9699691$$
   $$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 + 1 = 223092871$$
   $$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 + 1 = 6469693231$$

   (b) Would Euclid's proof have worked if instead of $2 \cdot 3 \cdots P + 1$ we had used

   $$N = 2 \cdot 3 \cdot 5 \cdot 7 \cdots P - 1$$

   or perhaps even $P! + 1$? As in the previous question, try factoring this sequence of numbers.

4. Is 221 prime? If not what are its factors?

5. What is the largest divisor you need to check to be sure that 397 is prime? How do you know it is the largest?

6. Is 8171 prime? How do you know for certain?

7. List all the primes between 1 and 500 using the *Sieve of Eratosthenes*.

Sieve of Eratosthenes

A *sieve* is like a strainer. We're going to strain out the primes.

To use this sieve, start with a long list of numbers (like the list below). Forget about 1, it is neither prime nor composite. You know that 2 is a prime, so circle it. Then, cross off all multiples of 2 $(4, 6, 8, 10, \ldots)$ since you know they cannot be prime. Then, circle the next number on the list that isn't crossed out (3) and circle it. Cross off all the multiple of that number and now repeat. (The next number to circle will be 5.) When you are done all the circled numbers will be prime.

Watch for patterns when you are doing all of this. It can be helpful to use different colored pencils.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110	111	112	113	114	115	116	117	118	119	120
121	122	123	124	125	126	127	128	129	130	131	132	133	134	135	136	137	138	139	140
141	142	143	144	145	146	147	148	149	150	151	152	153	154	155	156	157	158	159	160
161	162	163	164	165	166	167	168	169	170	171	172	173	174	175	176	177	178	179	180
181	182	183	184	185	186	187	188	189	190	191	192	193	194	195	196	197	198	199	200
201	202	203	204	205	206	207	208	209	210	211	212	213	214	215	216	217	218	219	220
221	222	223	224	225	226	227	228	229	230	231	232	233	234	235	236	237	238	239	240
241	242	243	244	245	246	247	248	249	250	251	252	253	254	255	256	257	258	259	260
261	262	263	264	265	266	267	268	269	270	271	272	273	274	275	276	277	278	279	280
281	282	283	284	285	286	287	288	289	290	291	292	293	294	295	296	297	298	299	300
301	302	303	304	305	306	307	308	309	310	311	312	313	314	315	316	317	318	319	320
321	322	323	324	325	326	327	328	329	330	331	332	333	334	335	336	337	338	339	340
341	342	343	344	345	346	347	348	349	350	351	352	353	354	355	356	357	358	359	360
361	362	363	364	365	366	367	368	369	370	371	372	373	374	375	376	377	378	379	380
381	382	383	384	385	386	387	388	389	390	391	392	393	394	395	396	397	398	399	400
401	402	403	404	405	406	407	408	409	410	411	412	413	414	415	416	417	418	419	420
421	422	423	424	425	426	427	428	429	430	431	432	433	434	435	436	437	438	439	440
441	442	443	444	445	446	447	448	449	450	451	452	453	454	455	456	457	458	459	460
461	462	463	464	465	466	467	468	469	470	471	472	473	474	475	476	477	478	479	480
481	482	483	484	485	486	487	488	489	490	491	492	493	494	495	496	497	498	499	500

8. The gaps between primes tends to get bigger as you go along. Lets see how much bigger this gaps gets.[1]

   (a) Between 1 and _____ , there are 10 primes.

   (b) Between 100 and _____ , there are 10 primes.

   (c) Between 400 and _____ , there are 10 primes.

   (d) The longest run of consecutive composite numbers between 1 and 100 starts with _____ and ends with _____ and is _____ long.

   (e) The longest run of consecutive composite numbers between 401 and 500 starts with _____ and ends with _____ and is _____ long.

9. Do the gaps between the primes get long and longer or is there eventually a record-size gap that never gets beaten?

   Factorials can help understand this problem. A factorial, like 5! means to multiply all the numbers between 1 and 8 or $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40320$.

   (a) Is the number below prime or composite? (Can you find a factor that is not 1 or itself?)

   $$10! + 2 = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 + 2$$

   (b) Is the number below prime or composite?

   $$10! + 3 = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 + 3$$

   (c) Based on your answers above, how many composite numbers must there be in a row, starting with $10! + 2$?

   (d) Name a number which you can be sure is in a string of 99 composite numbers.

   (e) Name a number which you can be sure is in a string of 999 composite numbers.

   (f) Explain how you can be sure any record of composite numbers is a row eventually gets broken.

10. At the beginning of your list there are three odd primes in a row: $3, 5, 7$.

    (a) Does this ever happen again in your list?

    (b) Can it ever happen later on, perhaps beyone 500? Why or why not?

11. You can find pairs of primes in your list that are two odd numbers in a row which are both prime. These are called *twin primes*, such as 17 and 19 or 41 and 43.

    (a) List all the twin primes less than 500.

    (b) We know there are infinitely many primes. We also know there is only one *prime triplet*. How many twin primes are there?

[1] This problem is alluding to a famous theorem, called the Prime Number Theorem, that states that the number of prime less than or equal to x is approximately equal to $\frac{x}{\ln x}$.

12. See what numbers can be made by adding two primes. For example

$$2+2=4, \quad 2+3=5, \quad 3+3=6, \quad 2+5=4, \quad 3+5=8, \quad 2+7$$

(a) Make all numbers less than 20 by adding two primes.

(b) Find a composite number that is not the sum of two primes. Hint: there is one in the 20s.

Is there such a number is the 30s? 40s? 50s?

(c) Find a pattern for such numbers. Hint: look at the size of the gaps between primes. Explain, based on your answer, exactly how you can tell whether an odd composite number is the sum of two primes or not.

(d) Can you find an even number which is not prime, and not the sum of two primes?[2]

13. A perfect square is what you get when you multiply an integer by itself.

(a) Write down the first 10 perfect sqaures.

(b) List all the prime numbers less than 100. For each prime, determine if it can be written as a sum of two perfect squares. For example, $5 = 4 + 1$ can be written as a sum but 7 can not.

For the numbers that can be written as a sum of squares, keep track of the squares you used.

(c) Find a pattern that tells you which primes can be written as the sum of two squares. Hint: look for where they fall in the sieve.

(d) Explain why the impossible numbers really are impossible.

(e) Explain why all the primes that are possible to write as a sum of squares really are possible (why does your pattern you found will work forever?). (Warning: difficult!)

14. **Prime Generating Functions:** Here we want to find a function (or formula) that gives only prime numbers and, ideally, all the prime numbers. Here are two such functions that don't quite work as expected.

[2]This is a famous conjecture called Goldbach's conjecture: *Every even integer n greater than two is the sum of two primes.* A similar statement (actually equivalent to Goldbach's conjecture) is *Every integer n greater than five is the sum of three primes.* The Goldbach conjecture has not been proved or disproved.

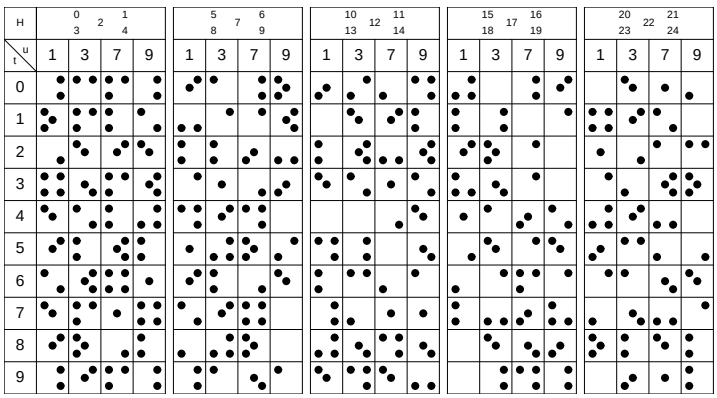
(a) Look at one less than powers of two. So, $2^1 - 1 = 1$ is not a prime. $2^2 - 1 = 3$ is prime.

n	$\mathbf{2^n - 1}$	**Is Prime?**
1	1	No
2	3	Yes
3		
4		
5		
6		
7		
8		
9		
10		
11		

Finish filling out the table, maybe even do some extras. See if there is a pattern for when $2^n - 1$ is prime and when it is not prime.

(b) Try this formula $n^2 - n + 41$ (due to Euler) and see if generates any primes.

n	$\mathbf{n^2 - n + 41}$	**Is Prime?**
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		
11		

How about negative numbers? How long will this pattern hold?

15. Here is a list of all the primes less than 2500 (graphic thanks to Paul Zeitz).