Math 132 - December 1, 2017
Solutions

Section 11.9: Functions as Power Series	
• Manupulating known power series (geometric): Substitution, Addition, Multiplication, Division (difficult), Differentiation, Integration	

1. **Clicker** (Method: multiplication)

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$$
$$\frac{1}{2+x} = \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \frac{x^4}{32} - \cdots$$

Use multiplication of power series to find a series for $\frac{1}{2+x} \cdot \ln(1+x)$.
(This is definitely a pain, but just distribute the terms out.)

$$\frac{1}{2+x} \cdot \ln(1+x) = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right)\left(\frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \frac{x^4}{32} - \cdots\right)$$

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{n+1}} x^n$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^{n+1}} x^n$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{n+1}} x^n$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{2^{n+1}} x^n$

(e) Something else **Correct**

Solution: It is VERY difficult to find a pattern when using the multiplication (or division) technique–don't even try!. Just multiply it out, distribute and get as many terms as you need.

$$\begin{aligned}\frac{1}{2+x} \cdot \ln(1+x) &= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right)\left(\frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \frac{x^4}{32} - \cdots\right) \\ &= \frac{x}{2} - \frac{x^2}{2} + \frac{5x^3}{12} - \frac{x^4}{3} + \frac{4x^5}{15} - \frac{13x^6}{60} + \frac{151x^7}{840} - \frac{16x^8}{105} + \frac{83x^9}{630} - \frac{73x^{10}}{630} + \cdots\end{aligned}$$

Note that the proposed answers are the following:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{n+1}} x^n = \frac{1}{4}x + \cdots$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^{n+1}} x^n = \frac{1}{4}x + \cdots$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{n+1}} x^n = \frac{1}{4}x + \cdots$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{2^{n+1}} x^n = \frac{2}{4}x - \frac{3}{8}x^2 + \cdots$

(e) Something else **Correct**

None of the answers can be correct!

2. (Method: Long Division–Challenging and worth avoiding whenever possible)
Use long division of power series to find a series for $\frac{\ln(1+x)}{\ln(1-x)}$
Note: $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \cdots$
and: $\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{5}x^5 - \cdots$

Solution: Tricky! This won't be tested but basically just use long division (I couldn't figure out a nice way to typeset the long division so no worked out solution here.)

$$\frac{\ln(1+x)}{\ln(1-x)} = -1 + x - \frac{1}{2}x^2 + \frac{5}{12}x^3 - \frac{7}{24}x^4 + \frac{191}{720}x^5 - \frac{33}{160}x^6 + \cdots$$

Lecture Notes: Integration of Power Series

Example: Power Series for

$$\begin{aligned}
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + x^4 + \cdots \\
\int \frac{1}{1-x}\,dx &= \int \left(1 + x + x^2 + x^3 + x^4 + \cdots\right)\,dx \\
-\ln(1-x) &= C + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots \\
\ln(1-x) &= -C - x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \cdots \\
\ln(1+x) &= -C + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots
\end{aligned}$$

To find C we plug in a "good" value of x to each side. How about $x = 0$? This gives $C = 0$. (Note: most of the time C is equal to 0, but DO NOT fall into the trap of assuming it is always 0!!)

This gives

$$\begin{aligned}
\ln(1+x) &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}x^n
\end{aligned}$$

Note that the radius of convergence of this power series is 1 and the interval of convergence is $(-1, 2]$.

3. (Method: Integration. Note $\int \frac{1}{1+x^2}\,dx = \arctan x + C$)
Find a series for $\arctan x$.

Solution:

$$\begin{aligned}
\arctan x &= \int \frac{1}{1+x^2}\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}x^{2n+1} \\
&= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots
\end{aligned}$$

Note, you can check to make sure C is correct by plugging in $x = 0$ on both sides of the equation. (This will definitely get you sometimes!)

4. Use your previous series to find a series for $\arctan(1) = \pi/4$

Solution: The series converges for $x = 1$ (why?).

$$\begin{aligned}\arctan 1 = \frac{\pi}{4} =& \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \\ =& 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots\end{aligned}$$

Convergence is terribly slow!

n	$4S_n$	$\lvert\pi - 4S_n\rvert$
10	3.232315809405592	0.0907
100	3.15149340107099	0.0099
1000	3.142591654339543	0.000999

5. Find a series for $\frac{1}{1+x^{10}}$ and use your series to approximate $\displaystyle\int_0^{0.25} \frac{1}{1+x^{10}}\, dx$

Solution:

$$\begin{aligned}\frac{1}{1+x} =& \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + x^4 - x^5 - x^6 + \cdots \\ \frac{1}{1+x^{10}} =& \sum_{n=0}^{\infty} (-1)^n x^{10n} = 1 - x^{10} + x^{20} - x^{30} + x^{40} - x^{50} - x^{60} + \cdots \\ \int \frac{1}{1+x^{10}}\, dx =& \sum_{n=0}^{\infty} \frac{(-1)^n}{10n+1} x^{10n+1} = x - \frac{1}{11}x^{11} + \frac{1}{21}x^{21} - \frac{1}{31}x^{31} + \frac{1}{41}x^{41} - \frac{1}{51}x^{50} - \frac{1}{61}x^{60} + \cdots \\ \int_0^{1/4} \frac{1}{1+x^{10}}\, dx =& \sum_{n=0}^{\infty} \frac{(-1)^n}{10n+1} \left(\frac{1}{4}\right)^{10n+1} \\ =& \frac{1}{4} - \frac{1}{11}\left(\frac{1}{4}\right)^{11} + \frac{1}{21}\left(\frac{1}{4}\right)^{21} - \cdots\end{aligned}$$

Note: no need to find C in the integral since we know we're going to plug in $x = 0.25$ and $x = 0$.
This is an alternating series. So, the integral is equal to $1/4$, accurate to $\frac{1}{11} \cdot \frac{1}{4^{11}} \approx 2 \times 10^{-8}$.

Section 11.10: Taylor Series

- Goal: Given a function: find a power series that equal the function
- Taylor Series
  $f(x) = \displaystyle\sum_{n=0}^{\infty} c_n (x-a)^n$ where $c_n = \frac{f^{(n)}(a)}{n!}$.
- Maclaurin Series (Taylor Series with $a = 0$)
  $f(x) = \displaystyle\sum_{n=0}^{\infty} c_n x^n$ where $c_n = \frac{f^{(n)}(0)}{n!}$.

6. Let $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

   (a) Find $f(0)$
   Solution: 1

   (b) Find $f(1)$
   Solution: You can compute partial sums to get an idea of this answer.

n	S_n
5	2.716666666666667
10	2.718281801146384
11	2.718281826198493
20	2.718281828459045
100	2.718281828459045
$e =$	2.718281828459045

In any case, $f(1) = e$, which isn't obvious yet.

(c) Find $f'(x) = f(x)$

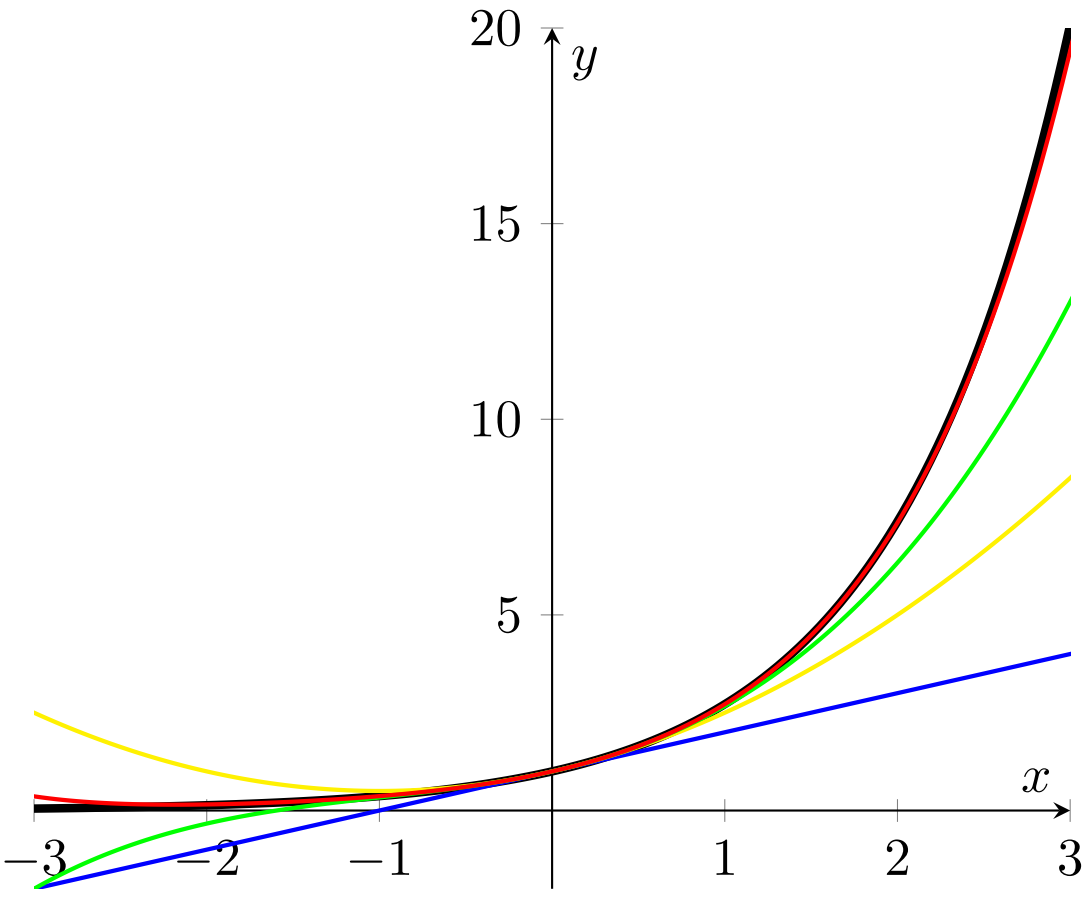
(d) Find $f''(x) = f(x)$

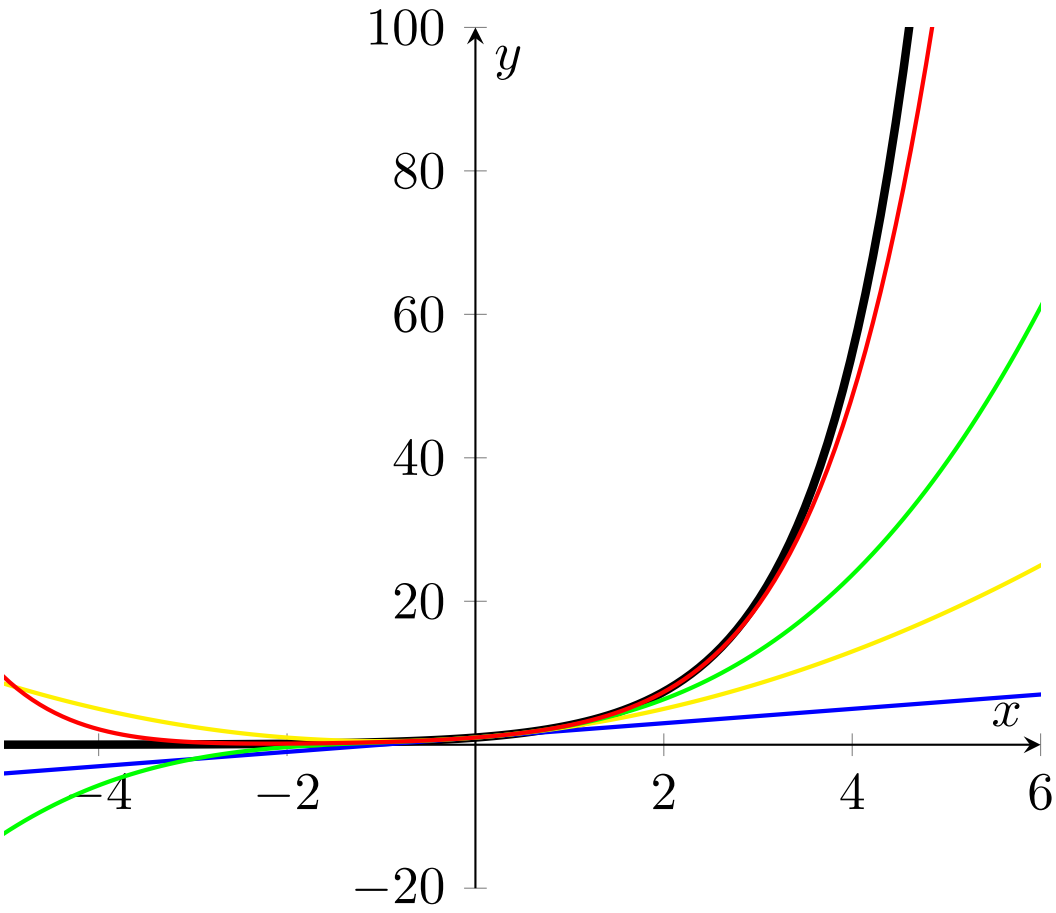
(e) Graph $f(x)$.

Lecture Notes: You can't (easily) graph $f(x)$ or answer this question without doing more work. One thing you know, is you're looking for a function $y = f(x)$ such that $\frac{dy}{dx} = y$ and $y(0) = 1$. This differential equation has a (unique) solution of $y = e^x$ and thus the series is equal to e^x. (This method isn't difficult, but it is beyond what we are doing in calc 2.)

Also, you can't easily graph a series, but you can graph partial sums, also called Taylor Polynomials.

$$f(x) = e^x \qquad \text{(Black)}$$
$$T_0 = 1$$
$$T_1 = 1 + x \qquad \text{(Blue)}$$
$$T_2 = 1 + x + \frac{x^2}{2!} \qquad \text{(Yellow)}$$
$$T_3 = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \qquad \text{(Green)}$$
$$T_4 = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$
$$T_5 = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$
$$T_6 = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} \qquad \text{(Red)}$$
$$T_7 = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!}$$
$$T_8 = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!}$$





Lecture Notes: The goal now is:

Given a function $f(x)$, find a representation of $f(x)$ with power series.

The idea is similar to:

- Let $f(x) = \frac{1}{1-x}$
- Then $f(x)$ can also be represented as $f(x) = 1 + x + x^2 + x^3 + x^4 + \cdots$.

Note: There will always be the question of radius of convergence, interval of convergence but these issue can confuse the main point above.

Idea of attack for a starting function $f(x)$.

- Suppose $f(x) = \sum_{n=0}^{\infty} c_n x^n$
- Plug in $x = 0$ to get $c_0 = f(0)$
- $\frac{d}{dx}$ gives $f'(x) = \sum_{n=1}^{\infty} c_n n x^{n-1}$
- Plug in $x = 0$ to get $c_1 = f'(0)$
- $\frac{d}{dx}$ gives $f''(x) = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2}$
- Plug in $x = 0$ to get $c_2 = f''(0)/2$
- Continue to get $c_n = \dfrac{f^{(n)}(0)}{n!}$

Do this for $f(x) = e^x$.

I suggest you set up a table to do this:

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$c_n = f^{(n)}(0)/n!$
0	e^x	1	1
1	e^x	1	1
2	e^x	1	$1/2$
3	e^x	1	$1/3!$
4	e^x	1	$1/4!$

Do this for $f(x) = \ln(1 + x)$

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$c_n = f^{(n)}(0)/n!$
0	$\ln(1+x)$	0	0
1	$1/(1+x)$	1	1
2	$-1/(1+x)^2$	-1	$1/2$
3	$2/(1+x)^3$	2	$2/3!$
4	$-6/(1+x)^4$	$-3!$	$-3!/4! = -1/4$
5	$4!/(1+x)^5$	$4!$	$4!/5! = 1/5$

This is called Taylor Series. Well, actually Maclaurin Series since we centered at $x = 0$ If we center at some point other than 0, it is Taylor Series centered at $x = a$.

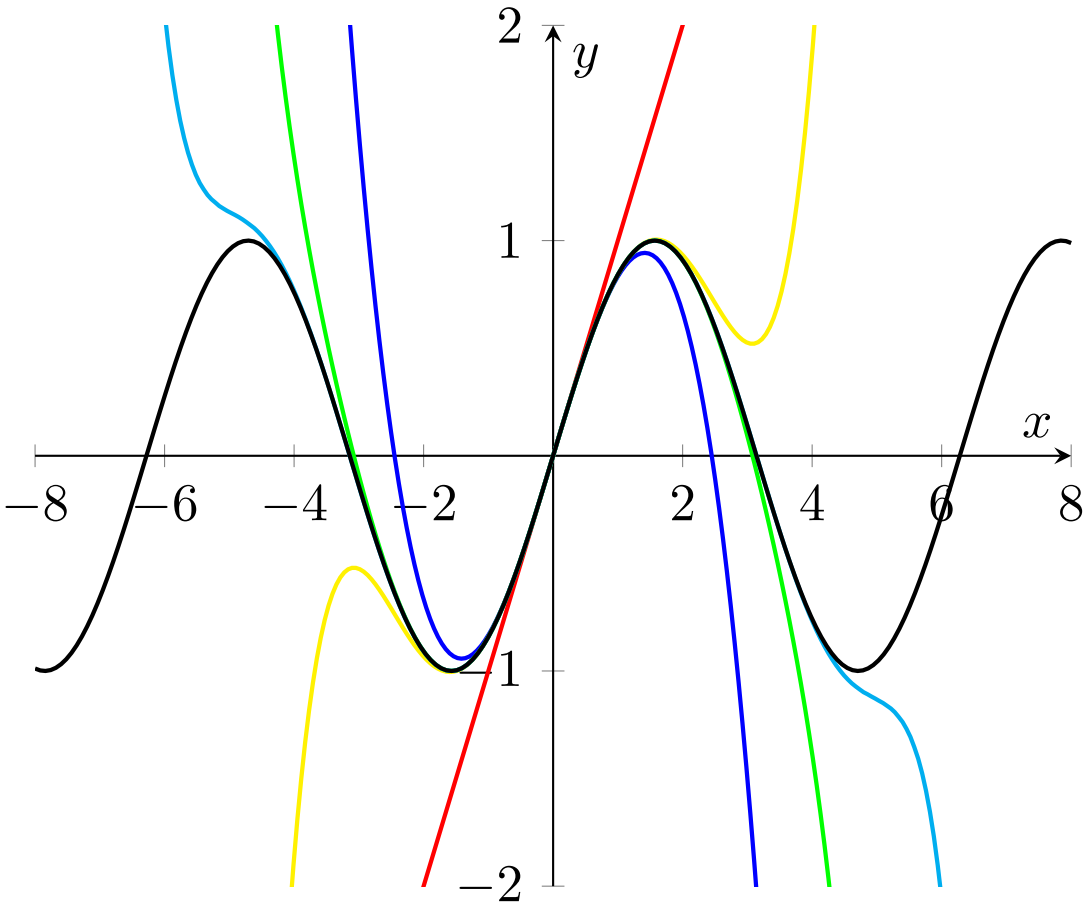
7. Find the Taylor Series for $f(x) = \sin x$ centered at $x = 0$
(Make a table for the derivatives!)

Solution:

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} + \cdots$$

Lets graph some of these Taylor Polynomials.

T_1: Red
T_3: Blue
T_5: Yellow
T_7: Green
T_{11}: Cyan
f: Black



8. Find the Taylor Series for $f(x) = \cos x$ centered at $x = 0$

Solution:

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^7}{7!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} + \cdots$$

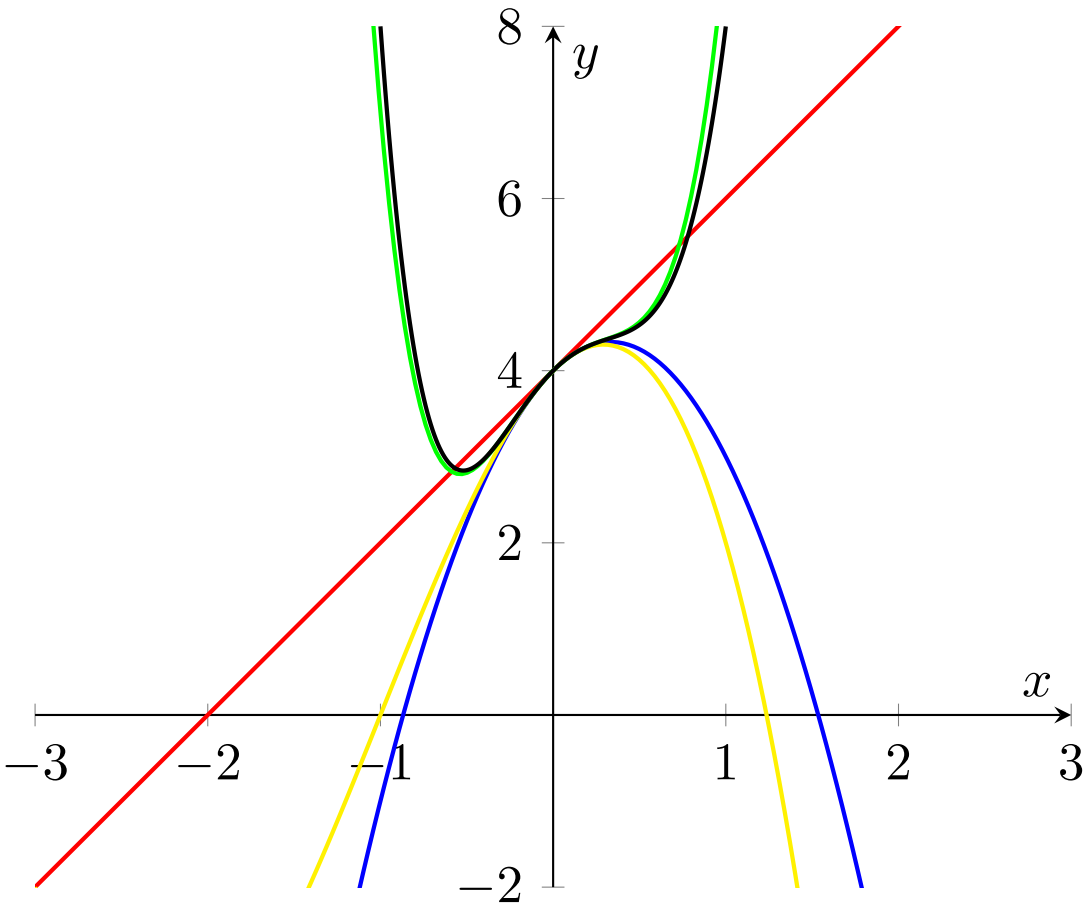
9. Find the Taylor Series for $f(x) = 4 + 2x - 3x^2 - x^3 + 7x^4 - x^5$ centered at $x = 0$.

Solution:

$$4 + 2\,x - 3\,x^2 - x^3 + 7\,x^4 - x^5$$

Lets graph some of these Taylor Polynomials.

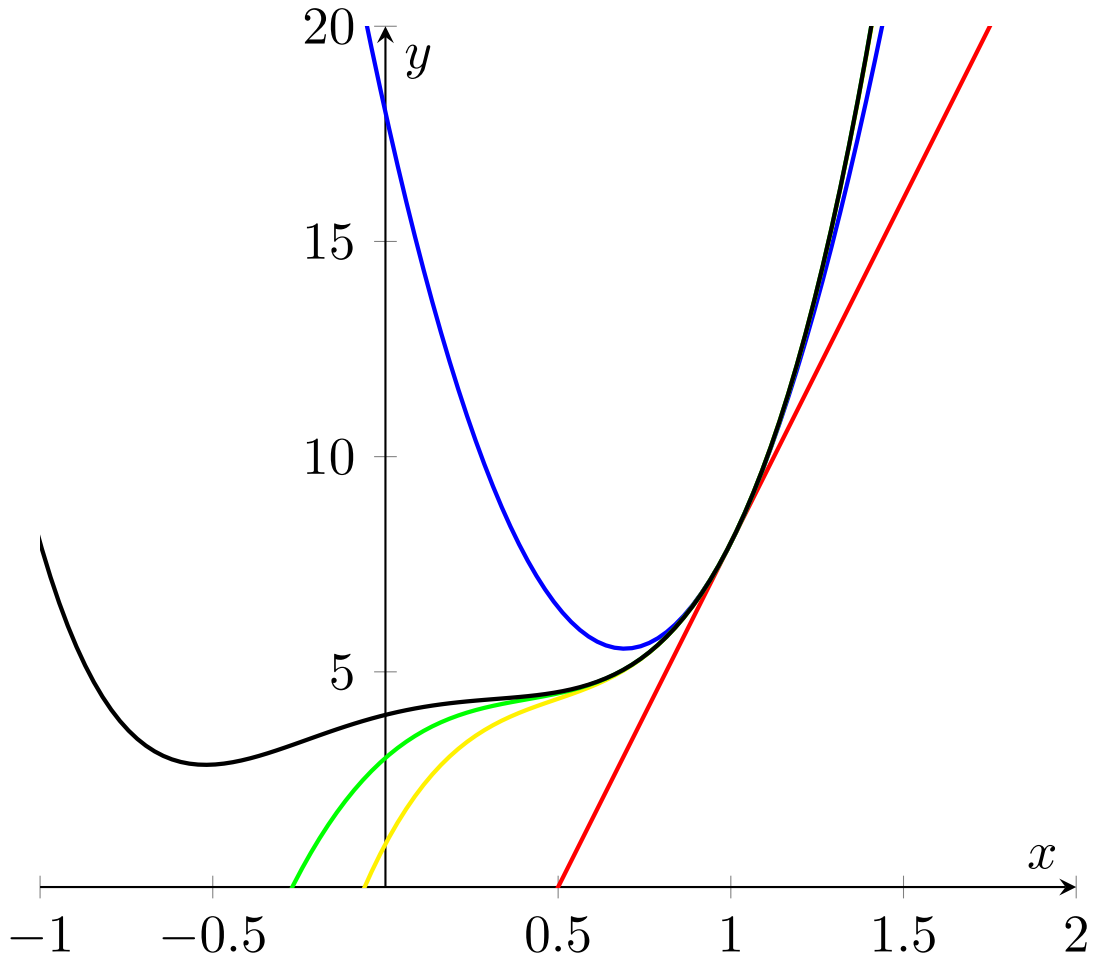
T_1: Red
T_2: Blue
T_3: Yellow
T_4: Green
$T_5 = f$: Black



10. Find the Taylor Series for $f(x) = 4 + 2x - 3x^2 - x^3 + 7x^4 - x^5$ centered at $x = 1$.

Solution:

$$8 + 16(x-1) + 26(x-1)^2 + 17(x-1)^3 + 2(x-1)^4 - (x-1)^5$$

Lets graph some of these Taylor Polynomials.

T_1: Red
T_2: Blue
T_3: Yellow
T_4: Green
$T_5 = f$: Black