Math 132: Discussion Session: Week 14

Directions: In groups of 3-4 students, work the problems on the following page. Below, list the members of your group and your answers to the specified questions. Turn **this paper** in at the end of class. You do not need to turn in the question page or your work.

Additional Instructions: It is okay if you do not completely finish all of the problems. Also, each group member should work through each problem, as similar problems may appear on the exam.

Scoring:

Correct answers	Grade
0–3	0%
4–6	80%
7–13	100%

Group Members:

11.7 Series Strategy.

(1) $\sum_{n=1}^{\infty} \frac{2+n}{1-2n}$ converges absolutely/converges conditionally/diverges. How do you know?

(2) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ converges absolutely/converges conditionally/diverges. How do you know?

(3) $\sum_{n=1}^{\infty} (\ln(n+1) - \ln n)$ converges absolutely/converges conditionally/diverges. How do you know?

(4) $\sum_{n=1}^{\infty} \sin^2\left(\frac{\pi}{n}\right)$ converges absolutely/converges conditionally/diverges. How do you know?

(5) $\sum_{n=1}^{\infty} \frac{n}{3^n - 2^n}$ converges absolutely/converges conditionally/diverges. How do you know?

(6) $\sum_{n=1}^{\infty} \frac{\cos n}{n^{\frac{3}{2}}}$ converges absolutely/converges conditionally/diverges. How do you know?

(7) $\sum_{n=1}^{\infty} (-1)^n (\ln(n+1) - \ln n)$ converges absolutely/converges conditionally/diverges. How do you know?

(8) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1.01}\ln(n+1)}$ converges absolutely/converges conditionally/diverges. How do you know?

(9) $\sum_{n=1}^{\infty} \frac{(-1)^n (2n)!}{(n!)^2}$ converges absolutely/converges conditionally/diverges. How do you know?

(10) $\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right)$ converges absolutely/converges conditionally/diverges. How do you know?

11.8: Power Series.

(1) $\sum_{n=1}^{\infty} nx^n$ converges when _____ $< x <$ _____.

(2) $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^3} x^n$ converges when _____ $< x <$ _____.

(3) $\sum_{n=2}^{\infty} \frac{x^n}{\ln n}$ converges when _____ $< x <$ _____.

Math 132 Discussion Session: Week 14

11.7 Series Strategy. Determine whether the following series converge absolutely, converge conditionally, or diverge, using any of the methods discussed so far in class. State the method you used, and how you used it. For example

- "$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-series test. The terms are already positive, so the series converges absolutely."
- "$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+2}$ converges absolutely by the comparison test because $\left|\frac{(-1)^n}{n^2+2}\right| = \frac{1}{n^2+2} \leq \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges."

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{2+n}{1-2n}$

Solution: Looking at the dominant terms of each sum, we see that

$$\lim_{n\to\infty} \frac{2+n}{1-2n} = \lim_{n\to\infty} \frac{n}{-2n} = -\frac{1}{2}.$$

Since the limit of the terms of the series is not zero, this series cannot converge. We're adding a lot of numbers that are close to $-\frac{1}{2}$, so the series must diverge to $-\infty$.

(2) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

Solution: This series is an alternating series. As $n \to \infty$, $\ln n$ gets bigger and bigger so $\lim_{n\to\infty} \frac{1}{\ln n} = 0$. Thus, the series converges by the alternating series test.

Does it converge absolutely? To find out, we need to test if

$$\sum_{n=2}^{\infty} \left|\frac{(-1)^n}{\ln n}\right| = \sum_{n=2}^{\infty} \frac{1}{\ln n}$$

converges.

Intuitively, $\ln n$ grows very slowly, so $\frac{1}{\ln n}$ goes to zero very slowly, and so the series might diverge. In fact, we know that $\ln n \leq n$, so

$$\frac{1}{\ln n} \geq \frac{1}{n},$$

and we know that $\frac{1}{n}$ goes to zero slowly enough that $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges. Thus, by the comparison test, the larger series $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ must also diverge. That means that the series does not converge absolutely.

We conclude that the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ converges conditionally.

(3) $\displaystyle\sum_{n=1}^{\infty} (\ln(n+1) - \ln n)$

Solution: This series is a telescoping series. Writing out the first few terms, we have that

$$\sum_{n=1}^{\infty} (\ln(n+1) - \ln n) = (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + (\ln 4 - \ln 3) + (\ln 5 - \ln 4) + \cdots.$$

Noting that $\ln 1 = 0$, the partial sums are

$$\begin{aligned} s_1 &= \ln 2, \\ s_2 &= \ln 3, \\ s_3 &= \ln 4, \\ s_4 &= \ln 5. \end{aligned}$$

Following the pattern, we see that $s_n = \ln(n+1)$. This sequence diverges to ∞. Since the sequence of partial sums diverges, the series diverges.

(4) $\displaystyle\sum_{n=1}^{\infty} \sin^2\left(\frac{\pi}{n}\right)$

Solution: It's worth a shot to see if the terms of the series might fail to converge to zero. As $n \to \infty$, we know that $\frac{\pi}{n} \to 0$, and $\sin 0 = 0$, so $\sin^2 \frac{\pi}{n} \to 0$. No luck, we'll have to try harder.

We want to do some sort of comparison test, and we know that $\sin^2 \frac{\pi}{n} \leq 1$, but that's not good enough. If we were to try to use that inequality, we'd note that $\sum_{n=1}^{\infty} 1$ diverges. So we know that our series is smaller than a divergent series, which tells us nothing.

We need a better approximation for $\sin \frac{\pi}{n}$. As n becomes large, $\frac{\pi}{n}$ becomes very small, and we have the approximation $\sin x \approx x$ for small x. In fact since $\sin x$ curves downwards, we know that $\sin x \leq x$ for all $x \geq 0$. Applying that fact, we see that

$$\sin^2\left(\frac{\pi}{n}\right) \leq \left(\frac{\pi}{n}\right)^2.$$

We know that $\sum_{n=1}^{\infty} \left(\frac{\pi}{n}\right)^2 = \pi^2 \sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges. Thus, the smaller series $\sum_{n=1}^{\infty} \sin^2\left(\frac{\pi}{n}\right)$ converges also. Since the terms are already positive, we know that this series converges absolutely.

(5) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n - 2^n}$

Solution: The dominant term in the denominator is 3^n, so we should try comparing this series to $\sum_{n=1}^{\infty} \frac{n}{3^n}$. To see if it's a valid comparison, the limit comparison test tells us to compute

$$\lim_{n\to\infty} \frac{\frac{n}{3^n}}{\frac{n}{3^n - 2^n}} = \lim_{n\to\infty} \frac{3^n - 2^n}{3^n} = \lim_{n\to\infty} \left(\frac{3^n}{3^n} - \frac{2^n}{3^n}\right) = \lim_{n\to\infty} \left(1 - \left(\frac{2}{3}\right)^n\right) = 1 - 0 = 1.$$

The limit exists and is above 0, so we've made a valid comparison, and to solve the problem what we need to do is see if the simpler series $\sum_{n=1}^{\infty} \frac{n}{3^n}$ converges.

There are several ways to do so. For example, the ratio test tells us to compute

$$\begin{aligned}\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{\frac{n+1}{3^{n+1}}}{\frac{n}{3^n}} \\ &= \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{3^n}{3^{n+1}} \\ &= 1 \cdot \frac{1}{3} = \frac{1}{3}.\end{aligned}$$

Since $\frac{1}{3} < 1$, the series $\sum_{n=1}^{\infty} \frac{n}{3^n}$ converges.

Alternatively, we can use the comparison test, noting that $n \leq 2^n$. The series $\sum_{n=1}^{\infty} \frac{2^n}{3^n}$ is a geometric series with common ratio $\frac{2}{3}$, so it converges. Thus the smaller series $\sum_{n=1}^{\infty} \frac{n}{3^n}$ also converges.

In any case, by the limit comparison test, since $\sum_{n=1}^{\infty} \frac{n}{3^n}$ converges, we conclude that $\sum_{n=1}^{\infty} \frac{n}{3^n - 2^n}$ converges. Since the terms are positive, we conclude that $\sum_{n=1}^{\infty} \frac{n}{3^n - 2^n}$ converges absolutely.

Note that the ordinary comparison test would not work because $\frac{n}{3^n - 2^n} > \frac{n}{3^n}$. The ordinary comparison test would tell us that $\sum_{n=1}^{\infty} \frac{n}{3^n - 2^n}$ is larger than a convergent series, which tells us nothing.

(6) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^{\frac{3}{2}}}$

Solution: The expression $\cos n$ is sometimes positive and sometimes negative, but it doesn't swap back and forth in a regular pattern, so the series is not alternating, and we cannot use the alternating series test.

But we can test for absolute convergence using the comparison test. We see that

$$\left|\frac{\cos n}{n^{3/2}}\right| \leq \frac{1}{n^{3/2}}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges by the p-series test, so the smaller series $\sum_{n=1}^{\infty} \left|\frac{\cos n}{n^{3/2}}\right|$ converges, and so the series $\sum_{n=1}^{\infty} \frac{\cos n}{n^{3/2}}$ converges absolutely.

(7) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\left(\ln(n+1)-\ln n\right)$

Solution: Note that $\ln(n+1)-\ln n$ is positive, so this is an alternating series. Using the alternating series test, we need to compute

$$\lim_{n\to\infty}\left(\ln(n+1)-\ln n\right)=\lim_{n\to\infty}\ln\left(\frac{n+1}{n}\right)=\ln 1=0.$$

Thus, the series converges by the alternating series test.

Does the series converge absolutely? Since $\ln(n+1)-\ln n$ is positive, we know that

$$\sum_{n=1}^{\infty}\left|(-1)^n\left(\ln(n+1)-\ln n\right)\right|=\sum_{n=1}^{\infty}\left(\ln(n+1)-\ln n\right).$$

We determined in a previous problem that $\sum_{n=1}^{\infty}\left(\ln(n+1)-\ln n\right)$ diverges. As a result, we know that the series $\sum_{n=1}^{\infty}(-1)^n(\ln(n+1)-\ln n)$ does not converge absolutely, so it converges conditionally.

(8) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{1.01}\ln(n+1)}$

Solution: This series is alternating. To see if it converges, we need to compute

$$\lim_{n\to\infty}\frac{1}{n^{1.01}\ln(n+1)}$$

As n becomes large, $n^{1.01}$ becomes large and $\ln(n+1)$ becomes large, so the denominator becomes large, and so the limit is zero. Thus, the series converges by the alternating series test.

To see if the series converges absolutely, we need to check if

$$\sum_{n=1}^{\infty}\frac{1}{n^{1.01}\ln(n+1)}$$

converges. We know that $\sum_{n=1}^{\infty}\frac{1}{n^{1.01}}$ converges by the p-series test, so, ideally, we can set up a comparison to something like $\sum_{n=1}^{\infty}\frac{1}{n^{1.01}}$ by writing an inequality for the $\ln(n+1)$ term. Since $n\geq 1$, we know that $\ln(n+1)\geq\ln 2$, so

$$\frac{1}{n^{1.01}\ln(n+1)}\leq\frac{1}{n^{1.01}\ln 2}$$

We know that $\sum_{n=1}^{\infty}\frac{1}{n^{1.01}\ln 2}=\frac{1}{\ln 2}\sum_{n=1}^{\infty}\frac{1}{n^{1.01}}$ converges. Thus, the smaller series $\sum_{n=1}^{\infty}\frac{1}{n^{1.01}\ln(n+1)}$ also converges, and so $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{1.01}\ln(n+1)}$ converges absolutely.

(9) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(2n)!}{(n!)^2}$

Solution: The ratio test works well with factorials. To apply the ratio test, we write down

$$|a_n|=\frac{(2n)!}{(n!)^2},\qquad\qquad |a_{n+1}|=\frac{(2(n+1))!}{((n+1)!)^2}.$$

Then we compute

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right|&=\frac{\frac{(2(n+1))!}{((n+1)!)^2}}{\frac{(2n)!}{(n!)^2}}\\&=\frac{(2(n+1))!}{(2n)!}\cdot\frac{(n!)^2}{((n+1)!)^2}\\&=\frac{(2n+2)!}{(2n)!}\cdot\left(\frac{n!}{(n+1)!}\right)^2\\&=(2n+2)(2n+1)\cdot\left(\frac{1}{n+1}\right)^2.\end{aligned}$$

To compute the limit of this expression, we look at the dominant terms.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}(2n)(2n)\left(\frac{1}{n}\right)^2 = 4.$$

Since this limit is greater than 1, the series diverges.

(10) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \sin\left(\frac{1}{n}\right)$

Solution: This is an alternating series, so we can see if it converges by computing

$$\lim_{n\to\infty}\sin\tfrac{1}{n}.$$

As $n\to\infty$, we know that $\frac{1}{n}\to 0$. Since $\sin 0 = 0$, we know that $\lim_{n\to\infty}\sin\frac{1}{n} = 0$. Thus, the series $\sum_{n=1}^{\infty}(-1)^n\sin\frac{1}{n}$ converges.

To find out whether it converges absolutely or conditionally, we need to see if

$$\sum_{n=1}^{\infty}\sin\tfrac{1}{n}$$

converges. There's no good way of working with sine, so we need to do a comparison test. As n becomes large, $\frac{1}{n}$ is very small, and for small numbers we know that $\sin x \approx x$. So, it's reasonable to compare $\sin\frac{1}{n}$ to $\frac{1}{n}$.

Unfortunately, the regular comparison test won't work here. We know that

$$\sin\tfrac{1}{n} \le \tfrac{1}{n}.$$

The series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, so the series $\sum_{n=1}^{\infty}\sin\frac{1}{n}$ is smaller than a divergent series, which tells us nothing.

However, the limit comparison test works just fine. We need to compute

$$\lim_{n\to\infty}\frac{\sin\frac{1}{n}}{\frac{1}{n}}.$$

To do so, we look at the corresponding function, see that we have an indeterminate form, and use L'Hôpital's rule. We find that

$$\lim_{x\to\infty}\frac{\sin\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty}\frac{\cos\frac{1}{x}\cdot-\frac{1}{x^2}}{-\frac{1}{x^2}} = \lim_{x\to\infty}\cos\tfrac{1}{x} = \cos 0 = 1.$$

Thus, $\lim_{n\to\infty}\frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1$. Since $0 < 1 < \infty$, we the limit comparison test works, so we can conclude from the fact that $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges that $\sum_{n=1}^{\infty}\sin\frac{1}{n}$ diverges also.

We conclude that the series $\sum_{n=1}^{\infty}(-1)^n\sin\frac{1}{n}$ converges conditionally.

11.8: Power Series. Find the range of values of x for which the following series converge. The ratio or root tests will be helpful, but after that you might need to check the endpoints of the range separately. In your answer, change the $<$ symbol to a $\le$ symbol when needed. If the range of values doesn't have a lower or upper bound, fill in the blanks with $-\infty$ and ∞.

(1) $\displaystyle\sum_{n=1}^{\infty} nx^n$

Solution: Let's use the ratio test. We see that $a_n = nx^n$ and $a_{n+1} = (n+1)x^{n+1}$. Thus

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)x^{n+1}}{nx^n}\right| \\ &= \lim_{n\to\infty}\frac{n+1}{n}\cdot|x| \\ &= 1\cdot|x| = |x|.\end{aligned}$$

The ratio test tells us that the series converges when $|x| < 1$ and diverges when $|x| > 1$.

When $|x| = 1$, we need to check the endpoints separately. When $x = 1$, the series is $\sum_{n=1}^{\infty} n$, which diverges because the terms go off to ∞. When $x = -1$, the series is $\sum_{n=1}^{\infty}(-1)^n n$, which diverges by the alternating series test.

Thus, the interval of convergence is $-1 < x < 1$, with the endpoints not included.

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^3} x^n$

Solution: To use the ratio test, we start by writing down

$$a_n = \frac{(2n)!}{(n!)^3} x^n, \qquad a_{n+1} = \frac{(2(n+1))!}{((n+1)!)^3} x^n.$$

Next, we need to compute

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{\frac{(2(n+1))!}{((n+1)!)^3} x^{n+1}}{\frac{(2n)!}{(n!)^3} x^n}\right| \\
&= \left|\frac{(2(n+1))!}{(2n)!} \cdot \frac{(n!)^3}{((n+1)!)^3} \cdot \frac{x^{n+1}}{x^n}\right| \\
&= \frac{(2n+2)!}{(2n)!} \cdot \left(\frac{n!}{(n+1)!}\right)^3 \cdot |x| \\
&= (2n+2)(2n+1) \cdot \left(\frac{1}{n+1}\right)^3 \cdot |x| .
\end{aligned}$$

To compute the limit, we look at the dominant terms in each sum

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} (2n)(2n) \cdot \left(\frac{1}{n}\right)^3 \cdot |x| = \lim_{n\to\infty} \frac{4}{n} \cdot |x| = 0.$$

No matter what x is, the limit is less than 1, so the series converges. Thus, the series converges for any number x. In the desired format, we write $-\infty < x < \infty$.

(3) $\displaystyle\sum_{n=2}^{\infty} \frac{x^n}{\ln n}$

Solution: To set up the ratio test, we write

$$a_n = \frac{x^n}{\ln n}, \qquad a_{n+1} = \frac{x^{n+1}}{\ln(n+1)}.$$

Next, we compute

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{\frac{x^{n+1}}{\ln(n+1)}}{\frac{x^n}{\ln n}}\right| \\
&= \left|\frac{\ln n}{\ln(n+1)} \cdot \frac{x^{n+1}}{x^n}\right| \\
&= \frac{\ln n}{\ln(n+1)} \cdot |x| .
\end{aligned}$$

We need to compute the limit of this expression. Intuitively $\ln n$ is very close to $\ln(n+1)$, so their ratio should be close to 1. We can check this intuition by writing the corresponding function and using L'Hôpital's rule. We have that

$$\lim_{x\to\infty} \frac{\ln x}{\ln(x+1)} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{x+1}} = \lim_{x\to\infty} \frac{x+1}{x} = 1.$$

Thus,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1 \cdot |x| = |x| .$$

Thus, we know that the series converges when $|x| < 1$ and diverges when $|x| > 1$. When $|x| = 1$, we need to check the endpoints separately.

When $x = 1$, the series is $\sum_{n=2}^{\infty} \frac{1}{\ln n}$. When $x = -1$, the series is $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$. We've considered both of these series earlier in the worksheet. The series $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges by comparison to $\sum_{n=2}^{\infty} \frac{1}{n}$. The series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ converges by the alternating series test.

Thus, the interval of convergence is $-1 \leq x < 1$, where we include one of the endpoints but not the other.