Math 132 - November 27, 2017
Solutions

Section 11.7: Series Strategy

Section 11.8: Power Series

- Goal: Apply what we've learned to series like these:

$$\sum_{n=0}^{\infty} a_n x^n \qquad \text{and} \qquad \sum_{n=0}^{\infty} a_n (x-a)^n$$

- For what values of x does a power series converge?
  - Radius of Convergence
  - Interval of Convergence
- For $\sum a_n(x-a)^n$, three possibilities.
  (i) The series converges for only $x = a$.
  (ii) The series converges for all x.
  (iii) There is a positive number R and series converges for $|x-a| < R$ but diverges for $|x-a| > R$.

1. **Clicker** Select the easiest x value to plug in to the power series $\displaystyle\sum_{n=0}^{\infty} x^n$

   (a) $x = 0$ **Correct** (b) $x = 1$ (c) $x = -1$ (d) No value is easy to plug in.

2. **Clicker** Select the easiest x value to plug in to the power series $\displaystyle\sum_{n=0}^{\infty} \frac{(x+1)^n}{n^2+1}$

   (a) $x = 0$ (b) $x = 1$ (c) $x = -1$ **Correct** (d) No value is easy to plug in.

Lecture Notes:
For a power series $\sum a_n(x-a)^n$, the point $x = a$ is call the center of the power series.

Lecture Notes: **Big Theorem!!**
For a given power series $\sum a_n(x-a)^n$, there are exactly three possibilities.

1. The series converges for only $x = a$.
2. The series converges for all x.
3. There is a positive number R and series converges for $|x-a| < R$ but diverges for $|x-a| > R$.

Note: in the third case, nothing is said for when $|x-a| = R$.

Lecture Notes:

Example: Find the Radius of Convergence and Interval of convergence of

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$$

Use the ratio test to find

$$L = \lim_{n\to\infty} \frac{|x|^{n+1}}{n+1} \cdot \frac{n}{|x|^n} = \lim_{n\to\infty} \frac{n|x|}{n+1} = |x|$$

In order of $L < 1$ we must have $|x| < 1$ and thus $R = 1$. The interval of convergence could be one of the following: $(-1,1)$, $[-1,1)$, $(-1,1]$, or $[-1,1]$. You have to test $x = -1$ and $x = 1$ to see which is correct. ($I = (-1,1]$).

3. Determine the Radius of Convergence and Interval of Convergence for the power series below.

$$\sum_{n=0}^{\infty} x^n$$

Solution: Radius: 1.
Interval: $(-1, 1)$.

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4. What is the difference between a **sequence** and a **series**?

   Solution: Sequence is just a list of terms. Series is a sum of terms

5. What does it mean for a sequence to **diverge**? **Converge?**

   Solution: Sequence converges if the terms eventually get close to one value.

   A sequence diverges if that doesn't happen.

6. What does it mean for a series to **diverge**? **Converge?**

   Solution: A series converges if it adds up to something.

   It diverges if it does not.

7. What is a **partial sum** and what are they good for?

   Solution: A partial sum is what you get when you stop adding more terms in a series. It is useful because if you go far enough out, you can get close to the sum of the series (provided the series converges).

8. Give an example of a convergent **geometric series**.

   Solution: $\sum 4\left(-\frac{99999}{100000}\right)^n$

9. Give an example of a divergent **geometric series**.

   Solution: $\sum 4\left(-\frac{99999}{99998}\right)^n$

10. What is the **Test for Divergence** and what is it good for?

    If $\lim a_n \neq 0$ then $\sum a_n$ must diverge.

11. What is the **Integral Test** and what is it good for?

    If $f(n) = a_n$ then either $\int_1^\infty f(x)\,dx$ and $\sum a_n$ both converge or both diverge.

12. What is a **p Series**?

    Solution: $\sum 1/n^p$

13. What is the **Comparison Test**?

    Solution:

    - If $0 \leq a_n \leq b_n$ and $\sum b_n$ converges then $\sum a_n$ converges.
    - If $0 \leq a_n \leq b_n$ and $\sum a_n$ diverges then $\sum b_n$ diverges.

14. What is the **Limit Comparison Test**?

    Solution: Let $L = \lim |a_n/b_n|$

    - If $\sum |b_n|$ converges and $L < \infty$ then $\sum a_n$ converges.
    - If $\sum |b_n|$ diverges and $L > 0$ then $\sum a_n$ diverges.

15. What is an **Alternating Series**?

    Solution: A series where every other term is a different sign.

16. What is the **Alternating Series Test**?

    Solution: $\sum (-1)^n b_n$ converges if $\lim b_n = 0$ and b_n decreasing.

17. What is **Conditionally Convergent**?

    $\sum a_n$ is conditionally convergent if $\sum a_n$ converges but $\sum |a_n|$ diverges.

18. What is the **Absolutely Convergent**?

$\sum a_n$ is absolutely convergent if $\sum |a_n|$ converges.

19. What is the **Ratio Test**?

Solution: Let $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

- If $L < 1$ then $\sum a_n$ converges absolutely
- If $L > 1$ then $\sum a_n$ diverges
- If $L = 1$ then Ratio Test is inconclusive

20. What is the **Root Test**?

Solution: Let $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$

- If $L < 1$ then $\sum a_n$ converges absolutely
- If $L > 1$ then $\sum a_n$ diverges
- If $L = 1$ then Root Test is inconclusive

21. **Clicker** Which converge:

I. $\displaystyle\sum_{n=2}^{\infty} \frac{n-1}{3n+1}$ II. $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

(a) Neither of them. (b) I only (c) II only **Correct** (d) I and II

Solution: I: Test for divergence. II: Ratio test

22. **Clicker** Which converge?

I. $\displaystyle\sum_{n=1}^{\infty} \left(2^{\frac{1}{n}} - 1\right)^n$ II. $\displaystyle\sum_{n=3}^{\infty} \frac{e^{\frac{1}{n}}}{n^2-3}$

(a) Neither of them. (b) I only (c) II only (d) I and II **Correct**

Solution: I: Root test. II: Limit Comparison to $1/n^2$.

23. **Clicker** Which converge?

I. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$ II. $\displaystyle\sum_{n=1}^{\infty} ne^{-n}$

(a) Neither of them. (b) I only (c) II only **Correct** (d) I and II

Solution: I: Compare to $1/n$. II: Ratio test.

24. **Clicker** Which converge conditionally, but not absolutely?

I. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n n!}{n^n}$ II. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^3}$

(a) Neither of them. **Correct** (b) I only (c) II only (d) I and II

Solution: I: Ratio test gives

$$\lim \frac{(n+1)!n^n}{(n+1)^{n+1}n!} = \lim \frac{n+1}{n+1}\left(\frac{n}{n+1}\right)^n = \lim \left(\frac{n}{n+1}\right)^n = \frac{1}{e}$$

II: Absolutely convergent by integral test.

25. **Clicker** Which converge?

I. $a_n = \dfrac{n^3 + \sqrt{3n^6 + 2n - 1}}{4n^3 + 2n + 5}$ II. $b_n = \dfrac{\cos\left(\frac{1}{n}\right)}{n+1}$

(a) Neither of them. (b) I only (c) II only (d) I and II **Correct**

Solution:

The key here is that this is talking about convergence of sequences.

I: converge to $(1 + \sqrt{3})/4$.

II: converges to 0

26. **Clicker** For which of the following series is the Ratio Test inconclusive?

I. $\displaystyle\sum_{n=2}^{\infty} \frac{n}{2^n}$ II. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^2}$

(a) Neither of them. (b) I only (c) II only **Correct** (d) I and II

Solution: I: II:

27. **Clicker** Which converge?

I. $\displaystyle\sum_{n=2}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$ II. $\displaystyle\sum_{n=1}^{\infty} \frac{\pi^{3n}}{n!}$

(a) Neither of them. (b) I only (c) II only (d) I and II **Correct**

Solution: I: Note $\lim a_n = 0$ but $\lim(a_n)^{1/n} = 1/e$ and thus converges by root test.

II: converges by Ratio test

28. **Clicker** Which converge?

I. $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n+1} - \sqrt{n-1}}{n}$ II. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n + n}{e^{3n} - 3}$

(a) Neither of them. (b) I only (c) II only (d) I and II **Correct**

Solution: I: Rationalize numerator to get $a_n = \frac{2}{n(\sqrt{n+1}+\sqrt{n-1})}$ and limit compare to $\frac{2}{n^{3/2}}$.

II: Limit compare to $1/e^{2n}$, converges.

29. Name the strategy(s) to use to determine convergence or divergence:

(a) $\sum (0.98)^n$

Solution: Geometric series

(b) $\sum (-5)^{-n}$

Solution: Geometric Series

(c) $\displaystyle\sum \frac{1}{n(n+6)}$

Solution: Telescoping and comparison to p series

(d) $\displaystyle\sum \frac{(-1)^n}{37n}$

Solution: Alternating Series test

(e) $\sum ne^{-n^2}$

Solution: Integral test or Ratio test

(f) $\displaystyle\sum \frac{2^n}{n!}$

Solution: Ratio test

(g) $\displaystyle\sum \frac{n!}{e^n}$

Solution: Ratio Test

(h) $\displaystyle\sum \frac{n^n}{3^{3n+1}}$

Solution: Root test