Math 132 - November 17, 2017
Solutions

Section 11.5: Alternating Series

Let $S = \sum(-1)^n b_n$ where $b_n > 0$ be an alternating series. S converges if the following are true:

- $b_{n+1} < b_n$ (so the b_n are decreasing)
- $\lim b_n = 0$

And $|R_n| = |S - S_n| \leq b_{n+1}$

Section 11.6: Absolute Convergence

- Definition: $\sum a_n$ is **absolutely convergent** if $\sum |a_n|$ converges.
- Definition: $\sum a_n$ is **conditionally convergent** if $\sum a_n$ converges but $\sum |a_n|$ diverges.
- **Theorem:** If $\sum a_n$ is absolutely convergent then $\sum a_n$ is convergent.

1. **Clicker** **Alternating Series:** Does the series converge or diverge:
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots$$
(a) Converge **Correct** (b) Diverge

Solution: This is the example to keep in mind of a conditionally convergent series. It converges by the alternating series test and does not converge absolutely.

Lecture Notes:

Alternating Series: Here we are talking about series whose terms switch sign. The series $\sum_{n=1}^{\infty}(-1)^{n+1}/n$ is the typical example to keep in mind. You can graph the partial sums on a number line. Note:
$S_1 = 1$
$S_2 = 1 - \frac{1}{2}$
$S_3 = 1 - \frac{1}{2} + \frac{1}{3}$
$S_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$
etc.

[DRAW THESE ON A NUMBER LINE]– you should see that the sums converge to something.

To see this rigorously, you should write $s_n =$ and see that these partial sums converge.

Here's how it works in this case: To see why this works, look at the following

$$s_2 = 1 - \frac{1}{2} > 0$$
$$s_4 = s2 + \left(\frac{1}{3} - \frac{1}{4}\right) > s_2$$
$$s_6 = s4 + \left(\frac{1}{5} - \frac{1}{6}\right) > s_4$$
$$s_8 = s6 + \left(\frac{1}{7} - \frac{1}{8}\right) > s_6$$

and so on so that we have

$$s_2 < s_4 < s_6 < \ldots < s_{2n} < \ldots$$

But also:

$$s_2 = 1 - \frac{1}{2} < 1$$
$$s_4 = 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \frac{1}{4} < 1$$
$$s_6 = 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \frac{1}{6} < 1$$

Thus, $\{s_{2n}\}$ is an increasing, bounded sequence, and thus converges.

Similarly, $s_{2n+1} = s_{2n} + \frac{1}{n+1}$ and thus has the same limit as s_{2n} since $\frac{1}{n+1} \to 0$.

All the points of the alternating series test are contained here!

Notice that if you want to find the sum, then the remainder is always closer than the "next term":

$$|R_n| = |S - S_n| < |a_{n+1}|$$

In this case, suppose you added up:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{99} \approx 0.698172179$$

This is within $\frac{1}{100}$ of the actual sum. If you wanted to get within $0.0001 = \frac{1}{10000}$ of the acual sum, you would have to find:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{9999} \approx 0.69319718$$

Interesting Observation:

$$\begin{aligned} S_{100,000} &\approx 0.69314218058494 \\ \ln 2 &\approx 0.69314718055994 \\ \frac{1}{100,101} &\approx 1 \times 10^{-5} = 10 \times 10^{-6} \\ |S_{100,000} - \ln 2| &\approx 5 \times 10^{-6} \end{aligned}$$

The book does this for the general case and proves the following:

Alternating Series Test

Let $S = \sum(-1)^n b_n$ where $b_n > 0$ be an alternating series. S converges if the following are true:

- $b_{n+1} < b_n$ (so the b_n are decreasing)
- $\lim b_n = 0$

And $|R_n| = |S - S_n| \leq b_{n+1}$

2. **Clicker** Converge or Diverge? $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

If it converges, find the sum accurate to 0.01.

(a) Converge **Correct** (b) Diverge

Solution: The b_n terms are positive, have limit to 0 and decreasing. Thus it converges.

To find the sum note that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{1} - \frac{1}{1} + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} - \cdots$$

Note that $\frac{1}{120} < \frac{1}{100} = 0.01$ and thus, accurate to at least this amount we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \approx \frac{1}{1} - \frac{1}{1} + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} = \frac{3}{8} = 0.375$$

Note that

$$\sum_{n=0}^{100} \frac{(-1)^n}{n!} \approx 0.36787944117$$

and that this is accurate to within $\frac{1}{101!} \approx 1 \times 10^{-160}$.

Also note that $e^{-1} \approx 0.36787944117\ldots$. Hmmmmmmmm.

3. Converge or Diverge? $\frac{1}{4} - \frac{2}{5} + \frac{4}{6} - \frac{5}{7} + \cdots$

   If it converges, find the sum accurate to 0.01.

   Solution: Diverges

4. Converge or Diverge? $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \cdots$

   If it converges, find the sum accurate to 0.1.

   Solution: The b_n terms are positive, have limit to 0 and decreasing. Thus it converges.

   To find the sum note that

   $$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} - \cdots$$

   Note that $\frac{1}{\sqrt{100}} = 0.1$ and thus, accurate to at least this amount we have

   $$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}} \approx 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots + \frac{1}{\sqrt{99}} = 0.6550236395$$

   Note that

   $$\sum_{n=1}^{1000} \frac{(-1)^{n+1}}{\sqrt{n}} \approx 0.58909121$$

   and that this is accurate to within $\frac{1}{\sqrt{1001}} \approx 0.032$.

   Here's some more

   $$\sum_{n=1}^{1000} \frac{(-1)^{n+1}}{\sqrt{n}} \approx 0.58909121$$
   $$\sum_{n=1}^{10000} \frac{(-1)^{n+1}}{\sqrt{n}} \approx 0.599898768$$
   $$\sum_{n=1}^{10^7} \frac{(-1)^{n+1}}{\sqrt{n}} \approx 0.604741$$

   This series converges very slowly.

5. Let $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$.

   (a) $S_{15} = 0.8245417573\ldots$.
   How close does the alternating series remainder theorem guarantee S_{15} to be from S?
   Solution: Within $\frac{1}{16^2} \approx 0.00390625$

   (b) If you want the partial sum to be within 0.0001 of the sum, how many terms do you need to take?
   Solution: You want to solve $\frac{1}{(n+1)^2} \leq 0.0001$. Solving gives $n \geq 99$. In this case, $S_{99} \approx 0.822517533374\ldots$.

6. Let $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$.

   (a) $S_{15} = 0.901676118\ldots$.
   How close does the alternating series remainder theorem guarantee S_{15} to be from S?
   Solution: Within $\frac{1}{16^3} \approx 0.000244140624$

   (b) If you want the partial sum to be within 0.001 of the sum, how many terms do you need to take?
   Solution: You want to solve $\frac{1}{(n+1)^3} \leq 0.001$. Solving gives $n \geq 9$. In this case, $S_9 \approx 0.0, 2116476\ldots$.

Lecture Notes: Sometimes negative terms vs always positive terms.

Here are the main points of this idea:

1. If $\sum |a_n|$ converges then we say that the series $\sum a_n$ is **Absolutely Convergent**. Here's an example: $\sum \dfrac{(-1)^n}{n^2}$ is absolutely convergent.
2. If $\sum a_n$ converges but $\sum |a_n|$ diverges then the series $\sum a_n$ is **Conditionally Convergent.**
3. Theorem If $\sum a_n$ is absolutely convergent then $\sum a_n$ converges.

7. **Clicker** Does the series converge absolutely: $\displaystyle\sum_{i=1}^{\infty} \frac{(-1)^{n+1}}{n^2+2n+1}$

   (a) Yes **Correct** (b) No (c) Converges conditionally (d) Diverges (e) Diverges conditionally

   Solution: Converging absolutely means that the series $\sum |a_n|$ converges. So, we need to test the series

   $$\sum_{i=1}^{\infty} \frac{1}{n^2+2n+1}$$

   which converges by a comparison to $\sum \frac{1}{n^2}$. So the series converges absoltely and the series converges.

8. Are the following absolutely convergent, conditionally convergent, convergent?

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+5}$

   Solution: Not convergent–fails alternating convergence test: $\lim b_n \neq 0$.

   (b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n-3}\sqrt{n}}{n+4}$

   Conditionally convergent by alternating series test. To see b_n is decreasing, look at $f(x) = \frac{\sqrt{x}}{x+4}$. $f'(x) = \frac{4-x}{2\sqrt{x}(x+4)^2}$ and is negative when $x > 4$.

   (c) $\displaystyle\sum_{n=2}^{\infty} \frac{\cos(\pi n)}{\sqrt{n}}$

   Solution: Converges conditionally by alternating series test.

   (d) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \sin(\pi/n)$

   Solution: First convince yourself it is alternating. For $n > 0$, we have $0 < \pi/n < \pi$ and thus $\sin(\pi/n) > 0$. And, $\lim \sin(\pi/n) = 0$. Finally, for $n \geq 2$, $\sin(\pi/n)$ is decreasing. Thus the series converges.

   But, is the series $\sum \sin(\pi/n)$ convergent? This is more difficult but you can do a limit comparison to $\sum \frac{1}{n}$ to see that it divergses.

   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$

   Solution: $b_n = \frac{\ln n}{n}$ which has limit 0. Let $f(x) = \frac{\ln x}{x}$ and $f'(x) = \frac{1-\ln x}{x^2} < 0$ when $x > e$. Thus converges.

   It does not converge absolutely.

9. **Clicker** The series $\displaystyle\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$ converges (why?). What about the following:

   $$\sum_{n=0}^{\infty} n\left(\frac{2}{3}\right)^n, \quad \sum_{n=0}^{\infty} n^2\left(\frac{2}{3}\right)^n, \quad \sum_{n=0}^{\infty} n^3\left(\frac{2}{3}\right)^n, \quad \sum_{n=0}^{\infty} n^{10000}\left(\frac{2}{3}\right)^n$$

   (a) None converge

   (b) Some, not all converge

   (c) All converge

10. Let $\sum_{n=0}^{\infty} ar^n$ be a geometric series (so $a_n = ar^n$).

(a) Compute: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r$

(b) Compute: $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} a^{1/n} r = r$