# Math 132 - November 15, 2017
**Solutions**

### Section 11.4: Comparison Tests

- **Comparison Test For Convergence:** $|a_n| \leq |b_n|$, $\sum |b_n|$ converges then $\sum |a_n|$ converges.
- **Comparison Test For Divergence:** $|b_n| \leq |a_n|$, $\sum |b_n|$ diverges then $\sum |a_n|$ diverges.
- **Limit Comparison Test 1:** Let $L = \lim_{n\to\infty} \frac{a_n}{b_n}$. If $0 < L < \infty$ then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
- Not in textbook
  **Limit Comparison Test 2:** Let $L = \lim_{n\to\infty} \frac{a_n}{b_n}$. If $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
- Not in textbook
  **Limit Comparison Test 3:** Let $L = \lim_{n\to\infty} \frac{a_n}{b_n}$. If $L = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

### Section 11.5: Alternating Series

Let $S = \sum(-1)^n b_n$ where $b_n > 0$ be an alternating series. $S$ converges if the following are true:

- $b_{n+1} < b_n$ (so the $b_n$ are decreasing)
- $\lim b_n = 0$

And $|R_n| = |S - S_n| \leq b_{n+1}$

### Section 11.6: Absolute Convergence

- Definition: $\sum a_n$ is **absolutely convergent** if $\sum |a_n|$ converges.
- Definition: $\sum a_n$ is **conditionally convergent** if $\sum a_n$ converges but $\sum |a_n|$ diverges.
- **Theorem:** If $\sum a_n$ is absolutely convergent then $\sum a_n$ is convergent.

1. **Clicker** Fill in the box appropriately (consider $n$ to be a large integer):
   $n^2 + 2n + 1$ ☐ $3n^2$
   (a) $\leq$ (b) $\geq$ (c) Neither
   **Solution:** Note that since $2n \leq n^2$ and $1 \leq n^2$ we have $n^2 + 2n + 1 \leq n^2 + n^2 + n^2 = 3n^2$.

## Lecture Notes: Comparison Tests

Here we just compare a series we don't know to a series we do know. The basic idea is pretty simple but there is, as usual, some fussing to do with inequalities.

**Example:** Does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+3n+1}$ converge or diverge?

**Solution:** Just compare to something we know.

$$\frac{1}{n^2+3n+1} \quad \longleftrightarrow \quad \frac{1}{n^2}$$

Why do we pick $\frac{1}{n^2}$ to compare to? Because $n^2$ is the main term in the denominator. Since $\sum \frac{1}{n^2}$ converges, we expect Does $\sum \frac{1}{n^2+3n+1}$ to also converge. And, since we are wanting to show convergence, we want the following inequality to be true:

$$\frac{1}{n^2+3n+1} \stackrel{?}{\leq} \frac{1}{n^2}$$

If this inequality is true, we are in good shape because it shows that $\sum \frac{1}{n^2+3n+1}$ converges. Otherwise we have to do some fussing.

The equality is valid (why?) and thus

$$\sum_{n=1}^{\infty} \frac{1}{n^2+3n+1} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$$

and our series converges because $\sum \frac{1}{n^2}$ converges.

2. Converge or Diverge? (use comparison test)

   $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-2n-1}{n^4+2n+1}$ compared to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

   **Solution:** Comparison test works fine since $\dfrac{n^2-2n-1}{n^4+2n+1} < \dfrac{n^2}{n^4} = \dfrac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{n^2-2n-1}{n^4+2n+1}$ converges.

3. Converge or Diverge? (use comparison test)

   $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+2n+1}{n^4-2n-1}$ compared to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

   **Solution:** Comparison test doesn't work the following isn't true: $\dfrac{n^2+2n+1}{n^4-2n-1} < \dfrac{n^2}{n^4} = \dfrac{1}{n^2}$

   So, we can either do something to make the inequality correct or else do a limit comparison (more advanced, but more useful). Here's how we can make the inequality work.

   First, notice that for large values of $n$ we have the following inequalities:
   $2n < n^2$ (true for $n > 2$)
   $2n < n^2$ (true for $n > 1$)
   $-2n > -\frac{1}{3}n^4$ (true for $n > 2$)
   $-1 > -\frac{1}{3}n^4$ (true for $n > 2$)

   $$\frac{n^2+2n+1}{n^4-2n-1} \leq \frac{n^2+n^2+n^2}{n^4-\frac{1}{3}n^4-\frac{1}{3}n^4} = \frac{3n^2}{\frac{1}{3}n^4} = \frac{9}{n^2}$$

   and thus $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+2n+1}{n^4-2n-1}$ converges because $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

**Lecture Notes:** The problem with the comparison test is making the inequalities work well. What we really want to say is something like the following:

1. $\sum \frac{n^2+2n+1}{n^4-2n-1}$ should be compared to $\sum \frac{n^2}{n^4}$.
2. Thus $\sum \frac{n^2+2n+1}{n^4-2n-1}$ should converge if and only if $\sum \frac{n^2}{n^4}$ converges.
3. Since $\sum \frac{n^2}{n^4}$ converges, so does $\sum \frac{n^2+2n+1}{n^4-2n-1}$.

The **Limit Comparison Test** does exactly this. You decide what to compare to, see if your comparison is a good comparison. If your comparison is good (this is the test part), then your comparison says both converge or both diverge.

4. **Clicker** Let $L = \lim_{n\to\infty} \frac{a_n}{b_n}$.
If $L = 1$, which of the following do you expect:
(a) $a_n < b_n$ (b) $a_n = b_n$ **Correct** (c) $a_n > b_n$ (d) None of these
**Solution:** Well, you actually wouldn't expect exact equality, but for large $n$ you would expect $a_n$ and $b_n$ to be very close.

5. **Clicker** Let $L = \lim_{n\to\infty} \frac{a_n}{b_n}$.
If $L = 0$, which of the following do you expect:
(a) $a_n < b_n$ **Correct** (b) $a_n = b_n$ (c) $a_n > b_n$ (d) None of these

6. **Clicker** Let $L = \lim_{n\to\infty} \frac{a_n}{b_n}$.
If $L = \infty$, which of the following do you expect:
(a) $a_n < b_n$ (b) $a_n = b_n$ (c) $a_n > b_n$ **Correct** (d) None of these

## Lecture Notes:

The **Limit Comparison Test** says the following (assuming $a_n$ and $b_n$ are positive.

1. **Limit Comparison Test 1:** Let $L = \lim_{n\to\infty} \frac{a_n}{b_n}$.
If $0 < L < \infty$ then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
2. Not in textbook
**Limit Comparison Test 2:** Let $L = \lim_{n\to\infty} \frac{a_n}{b_n}$.
If $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
3. Not in textbook
**Limit Comparison Test 3:** Let $L = \lim_{n\to\infty} \frac{a_n}{b_n}$.
If $L = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

7. Converge or Diverge? (do a limit comparison test)
$$\sum_{n=1}^{\infty} \frac{n^2-2n-1}{n^4+2n+1} \text{ compared to } \sum_{n=1}^{\infty} \frac{1}{n^2}$$
**Solution:** $\lim \frac{a_n}{b_n} = \frac{\left(\frac{n^2-2n-1}{n^4+2n+1}\right)}{\left(\frac{1}{n^2}\right)} = 1$ which means that series behave the same and thus both converge.

8. Converge or Diverge? (do a limit comparison test)
$$\sum_{n=1}^{\infty} \frac{n^2+2n+1}{n^4+2n-1} \text{ compared to } \sum_{n=1}^{\infty} \frac{1}{n^2}$$
**Solution:** $\lim \frac{a_n}{b_n} = \frac{\left(\frac{n^2+2n+1}{n^4+2n-1}\right)}{\left(\frac{1}{n^2}\right)} = 1$ which means that series behave the same and thus both converge.

9. Determine convergence of the following. Use a comparison or limit comparison (try to use both!).

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+1}$

**Solution:** Diverges with a limit comparison to $\frac{1}{n}$. You can also do a "normal" comparison to $\frac{n^2}{2n^3}$ but that is much trickier.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$

**Solution:** Diverges with a limit comparison to $\frac{2n}{n^2} = \frac{2}{n}$.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n-1}$

**Solution:** Converges with a limit comparison to $\frac{1}{2^n}$.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^{3/2}}$

**Solution:** Converges with a limit comparison to $b_n = \frac{n^{1/4}}{n^{3/2}} = \frac{1}{n^{5/4}}$. Note, I choose $n^{1/4}$ by some trial and error. Note that $\lim \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, so $\sum a_n$ must converge too.

10. **Alternating Series:** Does the series converge or diverge:
$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots$

**Solution:** This is the example to keep in mind of a conditionally convergent series. It converges by the alternating series test and does not converge absolutely.

## Lecture Notes:

**Alternating Series:** Here we are talking about series whose terms switch sign. The series $\sum_{n=1}^{\infty}(-1)^{n+1}/n$ is the typical example to keep in mind. You can graph the partial sums on a number line. Note:
$S_1 = 1$
$S_2 = 1 - \frac{1}{2}$
$S_3 = 1 - \frac{1}{2} + \frac{1}{3}$
$S_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$
etc.

[DRAW THESE ON A NUMBER LINE]– you should see that the sums converge to something.

To see this rigorously, you should write $s_n =$ and see that these partial sums converge.

Here's how it works in this case: To see why this works, look at the following

$$s_2 = 1 - \frac{1}{2} > 0$$
$$s_4 = s2 + \left(\frac{1}{3} - \frac{1}{4}\right) > s_2$$
$$s_6 = s4 + \left(\frac{1}{5} - \frac{1}{6}\right) > s_4$$
$$s_8 = s6 + \left(\frac{1}{7} - \frac{1}{8}\right) > s_6$$

and so on so that we have

$$s_2 < s_4 < s_6 < \ldots < s_{2n} < \ldots$$

But also:

$$s_2 = 1 - \frac{1}{2} < 1$$
$$s_4 = 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \frac{1}{4} < 1$$
$$s_6 = 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \frac{1}{6} < 1$$

Thus, $\{s_{2n}\}$ is an increasing, bounded sequence, and thus converges.

Similarly, $s_{2n+1} = s_{2n} + \frac{1}{n+1}$ and thus has the same limit as $s_{2n}$ since $\frac{1}{n+1} \to 0$.

All the points of the alternating series test are contained here!

Notice that if you want to find the sum, then the remainder is always closer than the "next term":

$$|R_n| = |S - S_n| < |a_{n+1}|$$

In this case, suppose you added up:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{99} \approx 0.698172179$$

This is within $\frac{1}{100}$ of the actual sum. If you wanted to get within $0.0001 = \frac{1}{10000}$ of the acual sum, you would have to find:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{9999} \approx 0.69319718$$

**Interesting Observation:**

$$\begin{aligned} S_{100,000} &\approx 0.69314218058494 \\ \ln 2 &\approx 0.69314718055994 \\ \frac{1}{100,101} &\approx 1 \times 10^{-5} = 10 \times 10^{-6} \\ |S_{100,000} - \ln 2| &\approx 5 \times 10^{-6} \end{aligned}$$

The book does this for the general case and proves the following:

**Alternating Series Test**

Let $S = \sum(-1)^n b_n$ where $b_n > 0$ be an alternating series. $S$ converges if the following are true:

- $b_{n+1} < b_n$ (so the $b_n$ are decreasing)
- $\lim b_n = 0$

And $|R_n| = |S - S_n| \leq b_{n+1}$

11. **Clicker** Converge or Diverge? $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

If it converges, find the sum accurate to 0.01.

(a) Converge **Correct** (b) Diverge

**Solution:** The $b_n$ terms are positive, have limit to 0 and decreasing. Thus it converges.

To find the sum note that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{1} - \frac{1}{1} + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} - \cdots$$

Note that $\frac{1}{120} < \frac{1}{100} = 0.01$ and thus, accurate to at least this amount we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \approx \frac{1}{1} - \frac{1}{1} + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} = \frac{3}{8} = 0.375$$

Note that

$$\sum_{n=0}^{100} \frac{(-1)^n}{n!} \approx 0.36787944117$$

and that this is accurate to within $\frac{1}{101!} \approx 1 \times 10^{-160}$.

Also note that $e^{-1} \approx 0.36787944117\ldots$. Hmmmmmmmm.

12. Let $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$.

    (a) $S_{15} = 0.8245417573\ldots$.
    How close does the alternating series remainder theorem guarantee $S_{15}$ to be from $S$?
    **Solution:** Within $\frac{1}{16^2} \approx 0.00390625$

    (b) If you want the partial sum to be within 0.0001 of the sum, how many terms do you need to take?
    **Solution:** You want to solve $\frac{1}{(n+1)^2} \leq 0.0001$. Solving gives $n \geq 99$. In this case, $S_{99} \approx 0.822517533374\ldots$.

13. Let $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$.

    (a) $S_{15} = 0.901676118\ldots$.
    How close does the alternating series remainder theorem guarantee $S_{15}$ to be from $S$?
    **Solution:** Within $\frac{1}{16^3} \approx 0.000244140624$

    (b) If you want the partial sum to be within 0.001 of the sum, how many terms do you need to take?
    **Solution:** You want to solve $\frac{1}{(n+1)^3} \leq 0.001$. Solving gives $n \geq 9$. In this case, $S_9 \approx 0.0, 2116476\ldots$.

## Lecture Notes: Sometimes negative terms vs always positive terms.

Here are the main points of this idea:

1. If $\sum |a_n|$ converges then we say that the series $\sum a_n$ is **Absolutely Convergent**. Here's an example: $\sum \frac{(-1)^n}{n^2}$ is absolutely convergent.
2. If $\sum a_n$ converges but $\sum |a_n|$ diverges then the series $\sum a_n$ is **Conditionally Convergent.**
3. Theorem If $\sum a_n$ is absolutely convergent then $\sum a_n$ converges.

14. **Clicker** Does the series converge absolutely: $\displaystyle\sum_{i=1}^{\infty} \frac{(-1)^{n+1}}{n^2+2n+1}$

    (a) Yes **Correct** (b) No (c) It converges conditionally

    **Solution:** Converging absolutely means that the series $\sum |a_n|$ converges. So, we need to test the series

    $$\sum_{i=1}^{\infty} \frac{1}{n^2+2n+1}$$

    which converges by a comparison to $\sum \frac{1}{n^2}$. So the series converges absoltely and the series converges.

15. Are the following absolutely convergent? Are they conditionally convergent?

    (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+5}$
    **Solution:** Not convergent–fails alternating convergence test: $\lim b_n \neq 0$ and

    (b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n-3}\sqrt{n}}{n+4}$
    Conditionally convergent by alternating series test. To see $b_n$ decreating, look at $f(x) = \frac{\sqrt{x}}{x+4}$. $f'(x) = \frac{4-x}{2\sqrt{x}(x+4)^2}$ and is negative when $x > 4$.

    (c) $\displaystyle\sum_{n=2}^{\infty} \frac{\cos(\pi n)}{\sqrt{n}}$
    **Solution:** Converges conditionally by alternating series test.

    (d) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \sin(\pi/n)$
    **Solution:** First convince yourself it is alternating. For $n > 0$, we have $0 < \pi/n < \pi$ and thus $\sin(\pi/n) > 0$. And, $\lim \sin(\pi/n) = 0$. Finally, for $n \geq 2$, $\sin(\pi/n)$ is decreasing. Thus the series converges.
    But, is the series $\sum \sin(\pi/n)$ convergent? This is more difficult but you can do a limit comparison to $\sum \frac{1}{n}$ to see that it divergses.

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$

**Solution:** $b_n = \frac{\ln n}{n}$ which has limit 0. Let $f(x) = \frac{\ln x}{x}$ and $f'(x) = \frac{1-\ln x}{x^2} < 0$ when $x > e$. Thus converges. It does not converge absolutely.