Math 132: Discussion Session: Week 12

Directions: In groups of 3-4 students, work the problems on the following page. Below, list the members of your group and your answers to the specified questions. Turn **this paper** in at the end of class. You do not need to turn in the question page or your work.

Additional Instructions: It is okay if you do not completely finish all of the problems. Also, each group member should work through each problem, as similar problems may appear on the exam.

Scoring:

Correct answers	Grade
0–2	0%
3–4	80%
5–10	100%

Group Members:

11.2: Series.

(1) (a) $\displaystyle\sum_{n=1}^{\infty} \frac{2 \cdot 3^n}{(-4)^n} =$

(b) $\displaystyle\sum_{n=1}^{\infty} \left(e^{1/n} - e^{1/(n+1)}\right) =$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{6 \cdot 2^{2n-1}}{3^n} =$

(d) $\displaystyle\sum_{n=1}^{\infty} (\sin 100)^n =$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{n+2}{2n+5} =$

(f) $\displaystyle\sum_{n=2}^{\infty} \frac{3^n + n}{2^n + 4} =$

(2) (a) $\displaystyle\sum_{n=1}^{\infty} (-5)^n x^n =$

- converges whenever x is strictly between $\square$ and $\square$.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n}$

- converges whenever x is strictly between $\square$ and $\square$.

11.3: Integral Test.

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n^2+2)^{\frac{3}{5}}}$

- Test used? What are the conditions for using that test?

(2) For which values of p does $\displaystyle\sum_{n=2}^{\infty} \frac{n^2}{(n^3+1)^p}$ converge?

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11.2: Series.

(1) Determine whether the series converges or diverges. If it converges, determine its value.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2 \cdot 3^n}{(-4)^n}$

Solution: This looks like a geometric series. Expanding the summation, we see that

$$\sum_{n=1}^{\infty} \frac{2 \cdot 3^n}{(-4)^n} = \frac{2 \cdot 3}{-4} + \frac{2 \cdot 3^2}{(-4)^2} + \frac{2 \cdot 3^3}{(-4)^3} + \frac{2 \cdot 3^4}{(-4)^4} + \cdots$$

Factoring out the first term, we see that

$$\sum_{n=1}^{\infty} \frac{2 \cdot 3^n}{(-4)^n} = \frac{2 \cdot 3}{-4} \left(1 + \left(-\frac{3}{4}\right) + \left(-\frac{3}{4}\right)^2 + \left(-\frac{3}{4}\right)^3 + \cdots\right)$$

Since, $\left|-\frac{3}{4}\right| < 1$, we can sum the geometric series, so

$$\sum_{n=1}^{\infty} \frac{2 \cdot 3^n}{(-4)^n} = \frac{2 \cdot 3}{-4} \left(\frac{1}{1 - \left(-\frac{3}{4}\right)}\right) = \frac{2 \cdot 3}{-4} \left(\frac{4}{7}\right) = \boxed{-\frac{6}{7}}.$$

(b) $\displaystyle\sum_{n=1}^{\infty} \left(e^{1/n} - e^{1/(n+1)}\right)$

Solution: This looks like it might be a telescoping series. Expanding out the summation, we see that

$$\sum_{n=1}^{\infty} \left(e^{1/n} - e^{1/(n+1)}\right) = \left(e^1 - e^{1/2}\right) + \left(e^{1/2} - e^{1/3}\right) + \left(e^{1/3} - e^{1/4}\right) + \left(e^{1/4} - e^{1/5}\right) + \cdots$$

Indeed, there's going to be cancellation when we compute the partial sums. We compute that

$$\begin{aligned} s_1 &= e^1 - e^{1/2}, \\ s_2 &= e^1 - e^{1/3}, \\ s_3 &= e^1 - e^{1/4}, \\ s_4 &= e^1 - e^{1/5}. \end{aligned}$$

In general $s_n = e - e^{1/(n+1)}$. Our next task is to compute $\lim_{n\to\infty} s_n$. As n becomes very large, $1/(n+1)$ becomes very close to 0. Since the exponential function is continuous, we see that

$$\lim_{n\to\infty} \left(e - e^{1/(n+1)}\right) = e - e^0 = \boxed{e - 1}.$$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{6 \cdot 2^{2n-1}}{3^n}$

Solution: This looks like a geometric series. Expanding the summation, we see that

$$\sum_{n=1}^{\infty} \frac{6 \cdot 2^{2n-1}}{3^n} = \frac{6 \cdot 2^1}{3^1} + \frac{6 \cdot 2^3}{3^2} + \frac{6 \cdot 2^5}{3^3} + \frac{6 \cdot 2^7}{3^4} + \cdots.$$

Factoring out the first term, we see that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{6 \cdot 2^{2n-1}}{3^n} &= \frac{6 \cdot 2^1}{3^1} \left(1 + \frac{\cdot 2^2}{3} + \frac{2^4}{3^2} + \frac{2^6}{3^3} + \cdots\right) \\ &= 4 \left(1 + \frac{4}{3} + \left(\frac{4}{3}\right)^2 + \left(\frac{4}{3}\right)^3 + \cdots\right). \end{aligned}$$

Since $\left|\frac{4}{3}\right| > 1$, this geometric series $\boxed{\text{diverges}}$.

(d) $\displaystyle\sum_{n=1}^{\infty}(\sin 100)^n$

Solution: The key idea here is to not get scared of the $\sin 100$. It's just a number, about -0.506, though the exact value doesn't matter. Something like $\sum_{n=1}^{\infty}(-0.506)^n$ is clearly a geometric series. Expanding out the summation, we see that

$$\begin{aligned}\sum_{n=1}^{\infty}(\sin 100)^n &= \sin 100 + (\sin 100)^2 + (\sin 100)^3 + (\sin 100)^4 + \cdots \\ &= \sin 100\left(1 + \sin 100 + (\sin 100)^2 + (\sin 100)^3 + \cdots\right).\end{aligned}$$

We don't need to know the exact value of $\sin 100$, but we do need to know that $|\sin 100| < 1$ or else the series will diverge. Fortunately, we know that $|\sin x| \leq 1$. But the series would still diverge if $\sin 100$ were equal to 1 or -1. Is that possible? No. We know that $\sin x = \pm 1$ when $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots$. The number 100 is not on this list. Thus, indeed, $|\sin 100|$ is *strictly* smaller than 1, so the series converges to

$$\sum_{n=1}^{\infty}(\sin 100)^n = \sin 100\left(\frac{1}{1-\sin 100}\right) = \boxed{\frac{\sin 100}{1-\sin 100}} = \boxed{\frac{1}{1-\sin 100} - 1}.$$

(e) $\displaystyle\sum_{n=1}^{\infty}\frac{n+2}{2n+5}$

Solution: To understand the sequence $\frac{n+2}{2n+5}$, we look at the leading terms, $\frac{n}{2n} = \frac{1}{2}$, so we see that $\lim_{n\to\infty}\frac{n+2}{2n+5} = \frac{1}{2}$. If we add up a ton of numbers that are all very close to $\frac{1}{2}$, we'll get a very big result. Thus, the partial sums get bigger and bigger, so the series $\boxed{\text{diverges}}$.

(f) $\displaystyle\sum_{n=2}^{\infty}\frac{3^n+n}{2^n+4}$

Solution: Each term of the sequence $\frac{3^n+n}{2^n+4}$ is bigger than one. If we add up a ton of terms that are bigger than one, we'll get a huge result. The partial sums get bigger and bigger, so the series $\boxed{\text{diverges}}$.

(2) Find the values of x for which the series converges. Determine the value of the series for those values of x.

(a) $\displaystyle\sum_{n=1}^{\infty}(-5)^n x^n$

Solution: The key here is to not get scared of variables. This is a geometric series. Expanding out the sum, we see that

$$\begin{aligned}\sum_{n=1}^{\infty}(-5)^n x^n &= -5x + (-5)^2x^2 + (-5)^3x^3 + (-5)^4x^4 + \cdots \\ &= (-5x)\left(1 + (-5x) + (-5x)^2 + (-5x)^3 + \cdots\right).\end{aligned}$$

This geometric series converges as long as $|-5x| < 1$, which happens as long as $|x| < \frac{1}{5}$, that is $\boxed{-\frac{1}{5} < |x| < \frac{1}{5}}$. When that is the case, the series converges to

$$\sum_{n=1}^{\infty}(-5)^n x^n = (-5x)\frac{1}{1-(-5x)} = \boxed{-\frac{5x}{1+5x}}.$$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n}$

Solution: This is also a geometric series. Keep in mind that the summation starts at $n = 0$, which makes the problem easier. Expanding out the sum, we have

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n} &= 1 + \frac{x-2}{3} + \frac{(x-2)^2}{3^2} + \frac{(x-2)^3}{3^3} + \cdots \\ &= 1 + \left(\frac{x-2}{3}\right) + \left(\frac{x-2}{3}\right)^2 + \left(\frac{x-2}{3}\right)^3 + \cdots .\end{aligned}$$

This geometric series converges as long as $\left|\frac{x-2}{3}\right| < 1$, which can be simplified to $|x-2| < 3$, which means $\boxed{-1 < x < 5}$.

11.3: Integral Test.

(1) Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n^2+2)^{\frac{3}{5}}}$ converges or diverges. State any tests used and show that all conditions of the test are satisfied.

Solution:

We can use the integral test, because the function $\frac{1}{(x^2+2)^{3/5}}$ is continuous, positive, and decreasing when $x \geq 1$

So, answering the question amounts to figuring out whether or not

$$\int_1^{\infty} \frac{1}{(x^2+2)^{3/5}}\, dx$$

converges. We can do so using the comparison test for improper integrals. The function $\frac{1}{(x^2)^{3/5}}$ is very similar to the one we started with, but it's much easier to integrate. Since $x^2 < x^2 + 2$, we know that $(x^2)^{3/5} < (x^2+2)^{3/5}$, and so

$$\frac{1}{(x^2)^{3/5}} > \frac{1}{(x^2+2)^{3/5}}.$$

Next, we integrate

$$\begin{aligned}\int_1^{\infty} \frac{1}{(x^2)^{3/5}}\, dx &= \int_1^{\infty} x^{-6/5}\, dx \\ &= \lim_{t\to\infty} \int_1^t x^{-6/5}\, dx \\ &= \lim_{t\to\infty} (-5)x^{-1/5}\Big|_1^t \\ &= \lim_{t\to\infty} (-5)\left(t^{-1/5} - 1\right) \\ &= (-5)(0-1) = 5.\end{aligned}$$

Thus, $\int_1^{\infty} \frac{1}{(x^2)^{3/5}}\, dx$ converges. Since $\frac{1}{(x^2+2)^{3/5}}$ is a smaller function, we know that $\int_1^{\infty} \frac{1}{(x^2+2)^{3/5}}\, dx$ also converges. By the integral test, we know that the corresponding series, $\sum_{n=1}^{\infty} \frac{1}{(n^2+2)^{3/5}}$ $\boxed{\text{converges}}$ also.

(2) For which values of p does $\displaystyle\sum_{n=2}^{\infty} \frac{n^2}{(n^3+1)^p}$ converge?

Solution: The first step is not be scared of variables. If it soothes you, pretend $p = 17$, solve the problem, and then stop pretending.

We'd like to apply the integral test, so we need to compute

$$\int_2^\infty \frac{x^2}{(x^3+1)^p}\,dx.$$

We can do this integral with the substitution $u = x^3 + 1$, $du = 3x^2\,dx$. Peforming the substitution, we obtain

$$\int \frac{x^2}{(x^3+1)^p}\,dx = \tfrac{1}{3}\int \frac{1}{u^p}\,du.$$

To understand the bounds of integration, we see that when $x = 2$, $u = 9$, and when x becomes very large, u also becomes very large. Thus

$$\int_2^\infty \frac{x^2}{(x^3+1)^p}\,dx = \tfrac{1}{3}\int_9^\infty \frac{1}{u^p}\,du.$$

From the textbook section 7.8 on improper integrals, we know that the integral $\int_9^\infty \frac{1}{u^p}$ converges when $p > 1$ and diverges if $p \leq 1$. Thus, $\int_2^\infty \frac{x^2}{(x^3+1)^p}$ converges when $p > 1$ and diverges if $p \leq 1$. So the sum $\sum_{n=2}^\infty \frac{n^2}{(n^3+1)^p}$ $\boxed{\text{converges when } p > 1 \text{ and diverges when } p \leq 1.}$