Math 132 - November 3, 2017
Solutions

11.1 Sequences

Terms

- Increasing sequence
- Decreasing Sequence
- Monotonic Sequence
- Bounded Above Sequence
- Bounded Below Sequence
- Bounded Sequence

Monotonic Sequence Theorem:
Every bounded monotonic sequence converges.

Warm-up Problems

1. **Clicker** Let $\{a_n\}$ be the sequence such that $a_n = \frac{3}{n+5}$. Find a number M such that $a_n \leq M$ for all n.

   (a) 0 (b) $\frac{1}{2}$ **Correct** (c) 1 **Correct** (d) 100 **Correct** (e) Not possible to find such an M

   Solution: Here the question is "Is the sequence $\{a_n\}$ bounded above and what is that upper bound?"

Lecture Notes: Sequences

With sequences we mainly want to find their limits (as $n \to \infty$). Thus, we need some theorems that will help us comute limits. Most of what we'll use should have been learned in calc 1 and are listed in the top of the sheet yesterday.

The terms listed above are important because they will help us tell if a sequence converges or not.

- Bounded Above Sequence
- Bounded Below Sequence
- Bounded Sequence
- Increasing sequence
- Decreasing Sequence
- Monotonic Sequence

Here's why we care about these terms:

> Every bounded monotonic sequence converges.

And, this is one usual way to tell if a sequence converges–just check that it is bounded and monotonic. Here's an example.

$a_n = \frac{3}{n+5}$.

You can check that is it decreasing in two ways. First

$$a_{n+1} = \frac{3}{n+6} < \frac{3}{n+5} = a_n$$

And, if $f(x) = \frac{3}{x+5}$ then $f'(x) = -\frac{3}{(x+5)^2} < 0$

Show $a_{n+1} < a_n$ is usually either really simple or really difficult. If really difficult, then we usually do the derivative way.

And, you can see that a_n is bounded because a_n is decreasing, thus bounded above. a_n is always positive and thus bounded below by 0.

Thus this sequence converges by the theorem.

2. Write the first few terms of the sequence and then find $\lim_{n\to\infty} a_n$

(a) $a_n = \frac{\ln n}{n}$
Solution: $\lim a_n = 0$. (Use L'Hopital's Rule)

(b) $a_n = \sqrt[n]{n}$
Solution: $a_n = n^{1/n} = \exp(\ln n^{1/n}) = \exp\left(\frac{\ln n}{n}\right)$. Thus, from previous, $\lim a_n = e^0 = 1$

(c) Let $x > 0$ be fixed. Let $a_n = x^{1/n}$.
Solution: $\lim a_n = 1$

3. Write the first few terms of the sequence and then find $\lim_{n\to\infty} a_n$ where $a_n = x^n$.

Solution: This depends on x.
If $|x| < 1$ then $\lim_{n\to\infty} x^n = 0$
If $x = 1$ then $\lim_{n\to\infty} x^n = 1$
If $x = -1$ then $\lim_{n\to\infty} x^n$ diverges
If $|x| > 1$ then $\lim_{n\to\infty} x^n$ diverges

4. Let x be fixed and let $a_n = \left(1 + \frac{x}{n}\right)^n$. Find $\lim_{n\to\infty} a_n$.

Solution:

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \exp\left[\ln\left(1+\frac{x}{n}\right)^n\right] \\
&= \exp\left[n\ln\left(1+\frac{x}{n}\right)\right] \\
&= \exp\left[\frac{\ln\left(1+\frac{x}{n}\right)}{1/n}\right] \qquad \text{(Now apply L'Hopital)} \\
&= \exp\left[\frac{-x/(n^2(1+x/n))}{-1/n^2}\right] \\
&= \exp\left[\frac{x/((1+x/n))}{1}\right] \\
&= \exp\left[\frac{x}{1+\frac{x}{n}}\right] = e^x
\end{aligned}$$

5. Let x be fixed and let $a_n = \frac{x^n}{n!}$. Find $\lim_{n\to\infty} a_n$.

Solution: Technically, the squeeze theorem is needed since

$$-\frac{|x^n|}{n!} \le \frac{x^n}{n!} \le \frac{|x^n|}{n!}$$

For this, imagine a huge fixed x, so the numerator is exponential. But, there is some $M > |x|$ so that eventually (for $n > M$):

$$\begin{aligned}
\frac{|x|^n}{n!} &= \frac{|x|^n}{1\cdot 2\cdot 3\cdots M(M+1)(M+2)(M+3)\cdots n} \\
&= \frac{|x|^n}{M!\cdot(M+1)(M+2)(M+3)\cdots n} \\
&\le \frac{|x|^n}{M!M^{n-M}} = \frac{|x|^n M^M}{M!M^n} = \left(\frac{M^M}{M!}\right)\left(\frac{|x|}{M}\right)^n
\end{aligned}$$

Note that $\frac{M^M}{M!}$ is just a constant and that since $|x| < M$, $\frac{|x|}{M} < 1$ is just a constant too. So, by Number 3,

$$\begin{aligned}
\lim_{n\to\infty} \frac{|x|^n}{n!} &\le \lim_{n\to\infty} \left(\frac{M^M}{M!}\right)\left(\frac{|x|}{M}\right)^n \\
&= \left(\frac{M^M}{M!}\right)\lim_{n\to\infty}\left(\frac{|x|}{M}\right)^n = 0
\end{aligned}$$

6. Find $\lim_{n\to\infty} a_n$ where $a_n = \frac{n!}{n^n}$.

Solution: Note that

$$a_n = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots n}{n\cdot n\cdot n\cdot n\cdot n\cdots n} = \left(\frac{1}{n}\right)\left(\frac{2\cdot 3\cdot 4\cdot 5\cdots n}{n\cdot n\cdot n\cdot n\cdots n}\right) \le \left(\frac{1}{n}\right)\left(\frac{n\cdot n\cdot n\cdot n\cdots n}{n\cdot n\cdot n\cdot n\cdots n}\right) = \frac{1}{n}$$

Thus $0 \le \lim a_n \le \lim \frac{1}{n} = 0$.

7. Continued Fractions (Fun and mathematically important, but not important for calculus class).

   (a) Find the limit of the sequence:

   $a_1 = 1$, $a_2 = 1 + \dfrac{1}{1}$, $a_3 = 1 + \dfrac{1}{1 + \dfrac{1}{1}}$, $a_4 = 1 + \dfrac{1}{1 + \dfrac{1}{1 + \dfrac{1}{1}}}$, $a_5 = 1 + \dfrac{1}{1 + \dfrac{1}{1 + \dfrac{1}{1 + \dfrac{1}{1}}}}$, etc.

   Solution: Note that if $L = \lim a_n$ then $L = 1 + \frac{1}{L}$ and so $L^2 - L - 1 = 0$ which gives $L = \frac{1 \pm \sqrt{5}}{2}$ so L has to be equal to $\frac{1+\sqrt{5}}{2}$.

   (b) Find the limit of the sequence:

   $a_1 = 2$, $a_2 = 2 + \dfrac{1}{1}$, $a_3 = 2 + \dfrac{1}{1 + \dfrac{1}{2}}$, $a_4 = 2 + \dfrac{1}{1 + \dfrac{1}{2 + \dfrac{1}{1}}}$, $a_5 = 2 + \dfrac{1}{1 + \dfrac{1}{2 + \dfrac{1}{1 + \dfrac{1}{1}}}}$, $a_6 = 2 + \dfrac{1}{1 + \dfrac{1}{2 + \dfrac{1}{1 + \dfrac{1}{1 + \dfrac{1}{1}}}}}$,

   $a_7 = 2 + \dfrac{1}{1 + \dfrac{1}{2 + \dfrac{1}{1 + \dfrac{1}{1 + \dfrac{1}{4 + \dfrac{1}{1}}}}}}$, $a_8 = 2 + \dfrac{1}{1 + \dfrac{1}{2 + \dfrac{1}{1 + \dfrac{1}{1 + \dfrac{1}{4 + \dfrac{1}{1 + \dfrac{1}{1}}}}}}}$, etc.

   In this sequence, the "diagonal" of the fractions follows the pattern: $[2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1, \ldots]$
   Solution: Amazingly, $\lim a_n = e$. See:
   `https://arxiv.org/pdf/math/0601660`

Class Problems

8. For each of the following sequences, determine if it is bounded (above/below/both), monotonic/increasing/decreasing.

   (a) $a_n = \dfrac{1}{n}$
   Solution: Bounded and monotonic decreasing

   (b) $a_n = \dfrac{-1}{n}$
   Solution: Bounded and monotonic increasing

   (c) $a_n = \dfrac{(-1)^n}{n}$
   Solution: Bounded but not monotonic

   (d) $a_n = \dfrac{n+1}{n}$
   Solution: Bounded and monotonic decreasing

   (e) $a_n = \dfrac{n+1}{n+2}$
   Solution: Bounded and monotonic increasing

   (f) $a_n = n^2$
   Solution: Monotonic increasing but unbounded

   (g) $a_n = -n^2$
   Solution: Monotonic decreasing but unbounded

   (h) $a_n = (-1)^n n^2$
   Solution: Unbounded but not monotonic

9. Determine if the following sequence is monotonic and bounded: $a_n = \frac{5n-2}{7n+3}$.
   Solution: Increasing since $f'(n) = 29/(7n+3)^2 > 0$. Bounded since $0 < a_n < 5/7$.

10. Experiment with the following recursive sequences. In particular, write down a few tersm, determine (if possible): monotonicity (increasing/decreasing), bounded (above/below), convergence (converge/diverge) and find an explicit formula for the terms.

    (a) $a_1 = 4$, $a_n = a_{n-1} + 5$
    Solution: This is an *arithmetic* sequence.
    $4, 9, 14, 19, \ldots$.
    $a_n = -1 + 5n$
    Increasing, bounded below, unbounded above, diverges

    (b) $a_1 = 16$, $a_n = \frac{1}{2}a_{n-1}$
    Solution: This is a *geometric* sequence.
    $16, 8, 4, 2, 1, \frac{1}{2}, \ldots$
    $a_n = \frac{32}{2^n}$
    Decreasing, bounded, converges to 0.

    (c) $a_1 = 1$, $a_n = \frac{1}{2}(a_{n-1} + 6)$
    Solution: $1, 2.5, 4.75, 5.375, 5.6875, 5.84375, 5.921875, \ldots$.
    If $L = \lim a_n$ then $L = \frac{1}{2}(L + 6)$. Solving gives $L = 6$.
    This doesn't show that the limit exists. Rather, this shows that if there is a limit, then it has to be equal to 6. To see the limit exists, we use mathematical induction to show that the sequence is bounded and increasing.
    Bounded, $a_1 < 6$. If $a_n < 6$ then $a_{n+1} = \frac{1}{2}(a_n + 6) < \frac{1}{2}(6 + 6) = 6$
    Increasing, true that $a_2 > a_1$. Suppose we know that $a_{k+1} > a_k$. Then, adding six and dividing by 2 gives $\frac{1}{2}(a_{k+1} + 6) > \frac{1}{2}(a_k + 6)$, which is $a_{k+2} > a_{k+1}$.

    (d) $a_1 = 0$, $a_2 = 1$, $a_{n+1} = a_n + 2a_{n-1}$
    Solution: $0, 1, 1, 3, 5, 11, 21, 43, \ldots$
    Increasing, unbounded, diverges,

    (e) For the sequence from Probelm 10d, find $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n}$
    Solution: Note that this sequence is undef, $1, 3, \frac{5}{2}, \frac{11}{5}, \ldots$.
    And, $\dfrac{a_{n+1}}{a_n} = \dfrac{a_n + 2a_{n-1}}{a_n} = 1 + \dfrac{2}{a_n/a_{n-1}}$ Thus, if $L = \lim a_{n+1}/a_n$ then $L = 1 + 2/L$ and $L^2 = L + 2$. $L^2 - L - 2 = (L - 2)(L + 1)$ and we must have $L = 2$.
    (Can you find a way to make $L = -1$?)
    As with previous question, this only shows that if the limit exists, it must be equal to 2. To see that the limit exists, more work is needed (much more difficult!).

11. **Clicker** For a fixed r (real number) and fixed n (positive integer), use long division to simplify

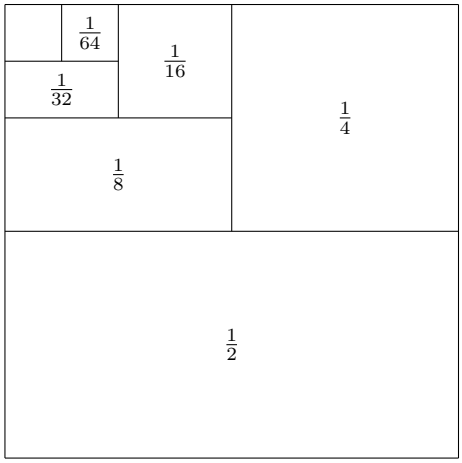
$$\frac{1 - r^n}{1 - r} = 1 + r + r^2 + \cdots + r^{n-1}$$

    (a) r^{n-1}
    (b) $1 + r^{n-1}$
    (c) $1 - r^{n-1}$
    (d) $1 + r + r^2 + r^3 + \cdots + r^{n-1}$ **Correct**

12. Find the limit of the following sequence: $\frac{1}{2}$,
    $\frac{1}{2} + \frac{1}{4}$,
    $\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$,
    $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$,
    $\ldots$
    The sequence is given by

$$s_n = \sum_{i=0}^{n} \left(\frac{1}{2}\right)^n$$

    Solution: First note that s_n is monotonic increasing. If it is also bounded then it converges. The following picture of a 1×1 square should convince you of all this. And, it should also convince you that the sum is equal to 1.


$\frac{1}{64}$
$\frac{1}{16}$
$\frac{1}{32}$
$\frac{1}{4}$
$\frac{1}{8}$
$\frac{1}{2}$