Math 132 - November 3, 2017

11.1 Sequences

Terms

- Increasing sequence
- Decreasing Sequence
- Monotonic Sequence
- Bounded Above Sequence
- Bounded Below Sequence
- Bounded Sequence

Monotonic Sequence Theorem:
Every bounded monotonic sequence converges.

Warm-up Problems

1. **Clicker** Let $\{a_n\}$ be the sequence such that $a_n = \frac{3}{n+5}$. Find a number M such that $a_n \leq M$ for all n.
   (a) 0 (b) $\frac{1}{2}$ (c) 1 (d) 100 (e) Not possible to find such an M
2. Write the first few terms of the sequence and then find $\lim_{n\to\infty} a_n$
   (a) $a_n = \frac{\ln n}{n}$
   (b) $a_n = \sqrt[n]{n}$
   (c) Let $x > 0$ be fixed. Let $a_n = x^{1/n}$.
3. Write the first few terms of the sequence and then find $\lim_{n\to\infty} a_n$ where $a_n = x^n$.
4. Let x be fixed and let $a_n = \left(1 + \frac{x}{n}\right)^n$. Find $\lim_{n\to\infty} a_n$.
5. Let x be fixed and let $a_n = \frac{x^n}{n!}$. Find $\lim_{n\to\infty} a_n$.
6. Find $\lim_{n\to\infty} a_n$ where $a_n = \frac{n!}{n^n}$.
7. Continued Fractions (Fun and mathematically important, but not important for calculus class).
   (a) Find the limit of the sequence:
   $$a_1 = 1,\ a_2 = 1 + \frac{1}{1},\ a_3 = 1 + \cfrac{1}{1 + \frac{1}{1}},\ a_4 = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{1}}},\ a_5 = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{1}}}},\ \text{etc.}$$
   (b) Find the limit of the sequence:
   $$a_1 = 2,\ a_2 = 2 + \frac{1}{1},\ a_3 = 2 + \cfrac{1}{1 + \frac{1}{2}},\ a_4 = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \frac{1}{1}}},\ a_5 = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \frac{1}{1}}}},\ a_6 = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{1}}}}},$$
   $$a_7 = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \frac{1}{1}}}}}},\ a_8 = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \frac{1}{1}}}}}}},\ \text{etc.}$$
   In this sequence, the "diagonal" of the fractions follows the pattern: $[2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1, \ldots]$

Class Problems

8. For each of the following sequences, determine if it is bounded (above/below/both), monotonic/increasing/decreasing.

   (a) $a_n = \dfrac{1}{n}$

   (b) $a_n = \dfrac{-1}{n}$

   (c) $a_n = \dfrac{(-1)^n}{n}$

   (d) $a_n = \dfrac{n+1}{n}$

   (e) $a_n = \dfrac{n+1}{n+2}$

   (f) $a_n = n^2$

   (g) $a_n = -n^2$

   (h) $a_n = (-1)^n n^2$

9. Determine if the following sequence is monotonic and bounded: $a_n = \frac{5n-2}{7n+3}$.

10. Experiment with the following recursive sequences. In particular, write down a few tersm, determine (if possible): monotonicity (increasing/decreasing), bounded (above/below), convergence (converge/diverge) and find an explicit formula for the terms.

    (a) $a_1 = 4$, $a_n = a_{n-1} + 5$

    (b) $a_1 = 16$, $a_n = \frac{1}{2} a_{n-1}$

    (c) $a_1 = 1$, $a_n = \frac{1}{2}(a_{n-1} + 6)$

    (d) $a_1 = 0$, $a_2 = 1$, $a_{n+1} = a_n + 2a_{n-1}$

    (e) For the sequence from Probelm 10d, find $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n}$

11. **Clicker** For a fixed r (real number) and fixed n (positive integer), use long division to simplify

    $$\frac{1-r^n}{1-r} =$$

    (a) r^{n-1}

    (b) $1 + r^{n-1}$

    (c) $1 - r^{n-1}$

    (d) $1 + r + r^2 + r^3 + \cdots + r^{n-1}$

12. Find the limit of the following sequence: $\frac{1}{2}$,
    $\frac{1}{2} + \frac{1}{4}$,
    $\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$,
    $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$,
    $\cdots$

    The sequence is given by

    $$s_n = \sum_{i=0}^{n} \left(\frac{1}{2}\right)^n$$