Math 132 - November 1, 2017
Solutions

4.4 L'Hopital's Rule

- $\left(\frac{0}{0}\right)$: If $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$ then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$
- $\left(\frac{\infty}{\infty}\right)$: If $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$ then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$

List of Indeterminate Forms:
$\frac{0}{0}$, $\frac{\infty}{\infty}$, $0\cdot\infty$, $\infty-\infty$, 0^0, 1^∞, ∞^0.
List of Indeterminate Forms handled by L'Hopital: $\frac{0}{0}$, $\frac{\infty}{\infty}$

11.1 Sequences

Terms

- Increasing sequence
- Decreasing Sequence
- Monotonic Sequence
- Bounded Above Sequence
- Bounded Below Sequence
- Bounded Sequence

Limit Properties

- $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$
- $\lim_{n\to\infty} c\cdot a_n = c\cdot \lim_{n\to\infty} a_n$
- $\lim_{n\to\infty} a_n\cdot b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$
- $\lim_{n\to\infty} c = c$
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$
- $\lim_{n\to\infty} a_n^p = \left(\lim_{n\to\infty} a_n\right)^p$
- If $\lim a_n = \lim c_n = L$ and $a_n < b_n < c_n$ then $\lim b_n = L$
- If $f(n) = a_n$ and $\lim_{x\to\infty} f(x) = L$ then $\lim a_n = L$

Warm-up Problems

1. **Clicker** Which limits below are indeterminant forms? Compute the limits using L'Hopital's Rule.

   (a) $\lim_{x\to 4} \frac{\sin x}{x}$

   (b) $\lim_{x\to 0} \frac{\sin x}{x}$ **Correct**

   (c) $\lim_{x\to 0} \frac{\cos x}{x}$

   (d) $\lim_{x\to\infty} xe^{-x}$ **Correct**

   Solution:

   (a) $\lim_{x\to 4} \frac{\sin x}{x}$: just plug in to get $\frac{\sin 4}{4}$.

   (b) $\lim_{x\to 0} \frac{\sin x}{x}$: this is indeterminant of type $0/0$

   $$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1$$

   (c) $\lim_{x\to 0} \frac{\cos x}{x}$: this is not indeterminant, but is of the form $1/0$ which is $\pm\infty$ (so limit does not exist).

   (d) $\lim_{x\to\infty} xe^{-x}$: this is indeterminant of the form $\infty\cdot 0$. To use L'Hopital's rule, you have to do some algebra first to put it in $0/0$ or ∞/∞ form.

   $$\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0$$

2. Set up arc length integral (compute if you have time!) for $y = \ln\cos(x)$ on $[0, \pi/4]$.

   Solution:

   $$L = \int_0^{\pi/4} \sqrt{1+\tan^2 x}\, dx = \ln(\sqrt{2}+1)$$

3. Set up surface area integral (compute if you have time!).
Rotate the arc $y = x^2$ between $x = 1$ to $x = 2$ about the line $x = -4$.

Solution:

$$\begin{aligned} SA &= \int 2\pi r \, ds \\ &= \int_1^2 2\pi(x+4)\sqrt{1+(2x)^2}\, dx \\ &= \int_1^2 2\pi x\sqrt{1+4x^2}\, dx + \int_1^2 2\pi 4\sqrt{1+4x^2}\, dx \approx 110.46 \end{aligned}$$

The first integral is trig substitution and the second is a simple substitution ($u = 1 + 4x^2$).

Or, you can do it dy:

$$\begin{aligned} SA &= \int 2\pi r \, ds \\ &= \int_1^4 2\pi(x+4)\sqrt{1+(dx/dy)^2}\, dy \\ &= \int_1^4 2\pi(\sqrt{y}+4)\sqrt{1+1/(4y)}\, dy = \int_1^4 \pi\left(1+\frac{4}{\sqrt{y}}\right)\sqrt{4y+1}\, dy \approx 110.46 \end{aligned}$$

Class Problems

Lecture Notes: A quick review of L'Hopital's rule. If you are struggling with this then you should review section 4.4 of the text. Here are the basic skills/procedures you should be familiar with for limits.

1. If you are given a limit, just plug in and see what you get. If its not indeterminant, then you probably have your answer.

   1.A. $\frac{1}{0}$ is either $+\infty$, $-\infty$ or both.

   1.B. $\frac{0}{1} = 0$

   1.C. For $\frac{0}{0}$ or $\frac{\infty}{\infty}$ use L'Hopital's Rule.

   1.D. For other indeterminant forms, use algebra tricks to convert it to $0/0$ or ∞/∞.

Example:

$$\lim_{x\to\infty} \frac{x+1}{2x-100}$$

This is of the form ∞/∞ and thus we can apply L'Hopital.

$$\lim_{x\to\infty} \frac{x+1}{2x-100} = \lim_{x\to\infty} \frac{1}{2} = \frac{1}{2}$$

The key is to identify the form first.
More examples below

4. Use L'Hopital's Rule, if necessary. Make sure you identify the form of the limit!

   (a) $\lim_{x\to 0} \frac{e^x - 1}{2x}$
   Solution: This is $0/0$ so L'Hopital's Rule applies

   $$\lim_{x\to 0} \frac{e^x - 1}{2x} = \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}$$

   (b) $\lim_{x\to\infty} \frac{e^x - 1}{2x}$
   Solution: This is ∞/∞ so L'Hopital's Rule applies

   $$\lim_{x\to\infty} \frac{e^x - 1}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

(c) $\lim_{x\to 0} \frac{e^x}{2x}$

Solution: This is $1/0$ so L'Hopital's Rule DOES NOT apply and the answer is that the limit does not exist $(\pm\infty)$.

(d) $\lim_{x\to \infty} \frac{e^x}{2x}$

Solution: This is ∞/∞ so L'Hopital's Rule applies

$$\lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

5. **Clicker** Find the next term in the sequence

$$1, 2, 3, \ldots$$

(a) 0 (b) 1 (c) 4 (d) 42 **Correct**

Solution: Using the formula

$$a_n \frac{(k-4)n^3 + (4-k)6n^2 + (11k-38)n + (-6k+24)}{6}$$

gives $a_1 = 1$, $a_2 = 2$, $a_3 = 3$ and $a_4 = k$. So, if we let $k = 42$ then we get our answer:

$$a_n = \frac{19n^3 - 114n^2 + 212n - 114}{3}$$

Note, this is an explicit definition of a sequence.

6. **Clicker** Find the next term in the sequence

$$1, 1, 2, 3, 5, 8, \ldots$$

(a) 9 (b) 10 (c) 11 (d) 12 (e) 13 **Correct**

Solution: This is the famous Fibonacci sequence $a_n = a_{n-1} + a_{n-2}$.

Note: This is a recursive definition of a sequence.

7. **Clicker** Find the next term in the sequence

$$157.41, \quad 163.05, \quad 166.72, \quad 168.97, \quad \ldots$$

(a) 151.12 (b) 163.32 (c) 171.21 (d) 175.21 (e) 181.98 **Correct**

Solution: These are stock prices for Apple over the previous week. Who knows where the price will be today but it would be great to know!

Lecture Notes:

The overview for this last chapter is the following:

1. Sequences of numbers.
   What does it mean for a sequence to converge?
2. Series: adding up infinite sequences.
   What does it mean for a series to converge?
3. Lots of ways to test if a series converges or diverges (find the actual sum is usually very difficult).
4. Using knowledge of series to write functions with series: Power Series

Lecture Notes: Sequences of numbers.

Here we have a sequence of numbers that we'll usually write as $\{a_n\}$ or $a_1, a_2, a_3, a_4, \ldots$.

The main thing we need to know how to do is take limits of these sequences (limits to ∞). Technically it is different than taking limits of functions but practically it is the same because if $f(n) = a_n$ then $\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$.

Here are some examples:

(a) $\lim_{n\to\infty} \frac{n}{n+1} = 1$

(b) $\lim_{n\to\infty} (-1)^n = DNE$

(c) $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$ (use squeeze theorem)

(d) $\lim_{n\to\infty} n! = \infty$

(e) $\lim_{n\to\infty} \frac{1}{n!} = 0$

8. For each of the sequences, write the first few terms of the sequence and then find $\lim_{n\to\infty} a_n$

   (a) $a_n = 2 + (0.1)^n$
   Solution: $\lim a_n = 2$

   (b) $a_n = \frac{n+(-1)^n}{n}$
   Solution: $\lim a_n = 1$

   (c) $a_n = \frac{1-2n}{1+2n}$
   Solution: $\lim a_n = -1$

   (d) $a_n = \sin n$
   Solution: Diverges

   (e) $a_n = \sin(1/n)$
   Solution: $\lim a_n = 0$

   (f) $a_n = \cos(1/n)$
   Solution: $\lim a_n = 1$

9. Trickier sequences. Write the first few terms of the sequence and then find $\lim_{n\to\infty} a_n$

   (a) $a_n = \frac{\ln n}{n}$
   Solution: $\lim a_n = 0$. (Use L'Hopital's Rule)

   (b) $a_n = \sqrt[n]{n}$
   Solution: $a_n = n^{1/n} = \exp(\ln n^{1/n}) = \exp\left(\frac{\ln n}{n}\right)$. Thus, from previous, $\lim a_n = e^0 = 1$

   (c) Let $x > 0$ be fixed. Let $a_n = x^{1/n}$.
   Solution: $\lim a_n = 1$

   (d) Let x be fixed and let $a_n = x^n$.
   Solution: This depends on x.
   If $|x| < 1$ then $\lim_{n\to\infty} x^n = 0$
   If $x = 1$ then $\lim_{n\to\infty} x^n = 1$
   If $x = -1$ then $\lim_{n\to\infty} x^n$ diverges
   If $|x| > 1$ then $\lim_{n\to\infty} x^n$ diverges

   (e) Let x be fixed and let $a_n = \left(1 + \frac{x}{n}\right)^n$.
   Solution:

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \exp\left[\ln\left(1+\frac{x}{n}\right)^n\right] \\
&= \exp\left[n\ln\left(1+\frac{x}{n}\right)\right] \\
&= \exp\left[\frac{\ln\left(1+\frac{x}{n}\right)}{1/n}\right] \qquad \text{(Now apply L'Hopital)} \\
&= \exp\left[\frac{-x/(n^2(1+x/n))}{-1/n^2}\right] \\
&= \exp\left[\frac{x/((1+x/n))}{1}\right] \\
&= \exp\left[\frac{x}{1+\frac{x}{n}}\right] = e^x
\end{aligned}$$

(f) Let x be fixed and let $a_n = \frac{x^n}{n!}$.

Solution: Technically, the squeeze theorem is needed since

$$-\frac{|x^n|}{n!} \leq \frac{x^n}{n!} \leq \frac{|x^n|}{n!}$$

For this, imagine a huge fixed x, so the numerator is exponential. But, there is some $M > |x|$ so that eventually (for $n > M$):

$$\begin{aligned}\frac{|x|^n}{n!} &= \frac{|x|^n}{1 \cdot 2 \cdot 3 \cdots M(M+1)(M+2)(M+3)\cdots n} \\ &= \frac{|x|^n}{M! \cdot (M+1)(M+2)(M+3)\cdots n} \\ &\leq \frac{|x|^n}{M! M^{n-M}} = \frac{|x|^n M^M}{M! M^n} = \left(\frac{M^M}{M!}\right)\left(\frac{|x|}{M}\right)^n\end{aligned}$$

Note that $\frac{M^M}{M!}$ is just a constant and that since $|x| < M$, $\frac{|x|}{M} < 1$ is just a constant too. So, by Number 9d,

$$\begin{aligned}\lim_{n\to\infty} \frac{|x|^n}{n!} &\leq \lim_{n\to\infty} \left(\frac{M^M}{M!}\right)\left(\frac{|x|}{M}\right)^n \\ &= \left(\frac{M^M}{M!}\right) \lim_{n\to\infty} \left(\frac{|x|}{M}\right)^n = 0\end{aligned}$$

(g) $a_n = \frac{n!}{n^n}$

Solution: Note that

$$a_n = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots n}{n \cdot n \cdot n \cdot n \cdot n \cdots n} = \left(\frac{1}{n}\right)\left(\frac{2 \cdot 3 \cdot 4 \cdot 5 \cdots n}{n \cdot n \cdot n \cdot n \cdots n}\right) \leq \left(\frac{1}{n}\right)\left(\frac{n \cdot n \cdot n \cdot n \cdots n}{n \cdot n \cdot n \cdot n \cdots n}\right) = \frac{1}{n}$$

Thus $0 \leq \lim a_n \leq \lim \frac{1}{n} = 0$.