Math 132: Discussion Session: Week 10

Directions: In groups of 3-4 students, work the problems on the following page. Below, list the members of your group and your answers to the specified questions. Turn **this paper** in at the end of class. You do not need to turn in the question page or your work.

Additional Instructions: It is okay if you do not completely finish all of the problems. Also, each group member should work through each problem, as similar problems may appear on the exam.

Scoring:

Correct answers	Grade
0–3	0%
4–6	80%
7–11	100%

Group Members:

7.5: Strategy for Integration.

(1) $\displaystyle\int \frac{1+\cos x}{1-\cos x}\,dx =$

(2) $\displaystyle\int \frac{1}{\sqrt{x+4}+\sqrt{x}}\,dx =$

(3) $\displaystyle\int \frac{2x^2\,dx}{(x^2-1)^{3/2}} =$

7.8: Improper Integrals.

(1) $\displaystyle\int_0^\infty \frac{1}{(x-3)^2}\,dx =$

(2) $\displaystyle\int_0^\infty \cos(3x)\,dx =$

(3) $\displaystyle\int_2^3 \frac{1}{\sqrt{3-x}}\,dx =$

(4) $\displaystyle\int_e^\infty \frac{1}{x(\ln x)^3}\,dx =$

8.1: Arclength.

(1) The arclength is:

(2) The arclength is:

(3) The arclength is:

(4) The arclength is:

Math 132 Discussion Session: Week 10

7.5: Strategy for Integration. Compute the following integrals using any integration method you can.

(1) $\displaystyle\int \frac{1+\cos x}{1-\cos x}\,dx$

Solution: The hard part of the expression is that it has things like $1+\cos x$ and $1-\cos x$, but our trig techniques only deal with products or quotients of $\cos x$ and $\sin x$. One way to deal with that issue is to use double-angle formulas. We have

$$\int \frac{1+\cos x}{1-\cos x}\,dx = \int \frac{2\cos^2\left(\frac{x}{2}\right)}{2\sin^2\left(\frac{x}{2}\right)}\,dx = \int \cot^2\left(\tfrac{x}{2}\right)\,dx.$$

This integral is simpler than the one we've started with, so we've made progress. The textbook lists strategies for integrating expressions involving tangent and secant, and the strategy for integrating expressions involving cotangent and cosecant is the same. A key formula is $1+\cot^2\theta = \csc^2\theta$. Applying this formula to the situation at hand, we find that

$$\int \cot^2\left(\tfrac{x}{2}\right)\,dx = \int \left(\csc^2\tfrac{x}{2} - 1\right)\,dx.$$

We know how to integrate both $\csc^2$ and 1. We make the substitution $u = \frac{x}{2}$, $du = \frac{1}{2}\,dx$, so

$$\begin{aligned}\int \left(\csc^2\tfrac{x}{2} - 1\right)\,dx &= 2\int \left(\csc^2 u - 1\right)\,du \\ &= 2(-\cot u - u) + C \\ &= \boxed{-2\cot\left(\tfrac{x}{2}\right) - x + C.}\end{aligned}$$

An alternative strategy for getting rid of the $1-\cos x$ in the denominator is analogous to the strategy for rationalizing denominators for expressions like $1-\sqrt{2}$. We multiply both the numerator and the denominator by $1+\cos x$ to find that

$$\begin{aligned}\int \frac{1+\cos x}{1-\cos x}\,dx &= \int \frac{(1+\cos x)(1+\cos x)}{(1-\cos x)(1+\cos x)}\,dx \\ &= \int \frac{1+2\cos x+\cos^2 x}{1-\cos^2 x}\,dx \\ &= \int \frac{1+2\cos x+\cos^2 x}{\sin^2 x}\,dx \\ &= \int \left(\frac{1}{\sin^2 x} + 2\frac{\cos x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x}\right)\,dx \\ &= \int \csc^2 x\,dx + 2\int \frac{\cos x}{\sin^2 x}\,dx + \int \cot^2 x\,dx.\end{aligned}$$

All three of these integrals are ones in the form we're used to for trigonometric integrals. We can do all three of them. We know that $\int \csc^2 x\,dx = -\cot x + C$ from the table of integrals. We can do the second integral using the substitution $u = \sin x$, $du = \cos x\,dx$. We have

$$\begin{aligned}\int \frac{\cos x}{\sin^2 x}\,dx &= \int \frac{1}{u^2}\,du \\ &= -\frac{1}{u} + C \\ &= -\csc x + C.\end{aligned}$$

We approach the last integral the same way as before.

$$\int \cot^2 x\,dx = \int \left(\csc^2 x - 1\right)\,dx = -\cot x - x.$$

Putting everything together, we find that

$$\begin{aligned}\int \frac{1+\cos x}{1-\cos x}\,dx &= -\cot x - 2\csc x - \cot x - x + C \\ &= \boxed{-2\cot x - 2\csc x - x + C.}\end{aligned}$$

For both solutions, the key strategy is to have a good sense of which integrals we can do, namely, products of trig functions, and to use algebra and trig identities to put our integral into that form.

(2) $\displaystyle\int \frac{1}{\sqrt{x+4}+\sqrt{x}}\,dx$

Solution: A reasonable substitution to try would be $u = x+4$, $du = dx$, gives

$$\int \frac{1}{\sqrt{x+4}+\sqrt{x}}\,dx = \int \frac{1}{\sqrt{u}+\sqrt{u-4}}\,du.$$

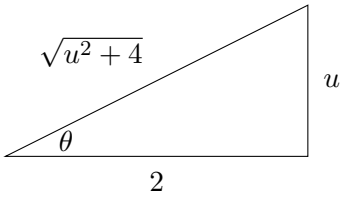
That's not any better. We could instead try the substitution $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}\,dx$. (The substitution $u = \sqrt{x+4}$ is also reasonable and would give about the same result.) We find that

$$\int \frac{1}{\sqrt{x+4}+\sqrt{x}}\,dx = \int \frac{1}{\sqrt{x+4}+\sqrt{x}} \cdot 2\sqrt{x} \cdot \frac{1}{2\sqrt{x}}\,dx.$$

To complete the substitution, we must be able to write $\sqrt{x+4}$ in terms of u. Since $u = \sqrt{x}$, we know that $x = u^2$, so $\sqrt{x+4} = \sqrt{u^2+4}$. Thus,

$$\int \frac{1}{\sqrt{x+4}+\sqrt{x}} \cdot 2\sqrt{x} \cdot \frac{1}{2\sqrt{x}}\,dx = \int \frac{1}{\sqrt{u^2+4}+u} \cdot 2u\,du.$$

The expression $\sqrt{u^2+4}$ means that we should do a trig substitution. The relevant triangle is



We see from the triangle that $u = 2\tan\theta$, so $du = 2\sec^2\theta\,d\theta$. We also see from the triangle that $\sqrt{u^2+4} = 2\sec\theta$. Thus,

$$\begin{aligned}\int \frac{1}{\sqrt{u^2+4}+u} \cdot 2u\,du &= \int \frac{1}{2\tan\theta + 2\sec\theta} \cdot 2 \cdot 2\tan\theta \cdot 2\sec^2\theta\,d\theta \\ &= 4\int \frac{\tan\theta\sec^2\theta}{\tan\theta+\sec\theta}\,d\theta.\end{aligned}$$

We could continue in this manner by multiplying the numerator and denominator by $\sec\theta - \tan\theta$ in order to get a denominator of $\sec^2\theta - \tan^2\theta = 1$.

$$\begin{aligned}4\int \frac{\tan\theta\sec^2\theta}{\tan\theta+\sec\theta}\,d\theta &= 4\int \frac{\tan\theta\sec^2\theta(\sec\theta-\tan\theta)}{\sec^2\theta-\tan^2\theta}\,d\theta \\ &= 4\int \left(\tan\theta\sec^3\theta - \tan^2\theta\sec^2\theta\right)\,d\theta.\end{aligned}$$

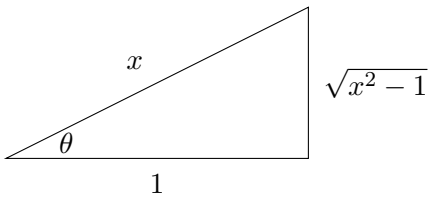
This is an integral we could do, but things have gotten quite complicated. Perhaps it's best to apply that idea to the original problem instead. Multiplying the numerator and denominator by $\sqrt{x+4}-\sqrt{x}$,

we find that

$$\begin{aligned}
\int \frac{1}{\sqrt{x+4}+\sqrt{x}}\,dx &= \int \frac{\sqrt{x+4}-\sqrt{x}}{(\sqrt{x+4}+\sqrt{x})(\sqrt{x+4}-\sqrt{x})}\,dx \\
&= \int \frac{\sqrt{x+4}-\sqrt{x}}{\sqrt{x+4}^2-\sqrt{x}^2}\,dx \\
&= \int \frac{\sqrt{x+4}-\sqrt{x}}{x+4-x}\,dx \\
&= \frac{1}{4}\int (\sqrt{x+4}-\sqrt{x})\,dx \\
&= \frac{1}{4}\left(\frac{2}{3}(x+4)^{3/2}-\frac{2}{3}x^{3/2}\right)+C \\
&= \boxed{\frac{1}{6}\left((x+4)^{3/2}-x^{3/2}\right)+C.}
\end{aligned}$$

(3) $\displaystyle\int \frac{2x^2\,dx}{(x^2-1)^{3/2}}$

Solution: The denominator is $\sqrt{x^2-1}^3$. For an expression of the form $\sqrt{x^2-1}$, we should use a trig substitution. The triangle we use is



From the triangle, we see that $\sqrt{x^2-1}=\tan\theta$ and $x=\sec\theta$, so $dx=\sec\theta\tan\theta$. We can then compute

$$\begin{aligned}
\int \frac{2x^2\,dx}{(x^2-1)^{3/2}} &= \int \frac{2\sec^2\theta(\sec\theta\tan\theta\,d\theta)}{\tan^3\theta} \\
&= 2\int \frac{\sec^3\theta}{\tan^2\theta}\,d\theta \\
&= 2\int \frac{1}{\sin^2\theta\cos\theta}\,d\theta.
\end{aligned}$$

At this point, trying either $u=\sin\theta$ or $u=\cos\theta$ would be reasonable. Trying $u=\sin\theta$, $du=\cos\theta\,d\theta$, we see that

$$\begin{aligned}
2\int \frac{1}{\sin^2\theta\cos\theta}\,d\theta &= 2\int \frac{1}{\sin^2\theta\cos\theta}\,d\theta \\
&= 2\int \frac{1}{\sin^2\theta\cos\theta}\frac{1}{\cos\theta}\cos\theta\,d\theta \\
&= 2\int \frac{1}{\sin^2\theta\cos^2\theta}\cos\theta\,d\theta.
\end{aligned}$$

To do the substitution, we must write the denominator in terms of $u=\sin\theta$, which we can do with the identity $\cos^2\theta=1-\sin^2\theta$. We find that

$$\begin{aligned}
2\int \frac{1}{\sin^2\theta\cos^2\theta}\cos\theta\,d\theta &= 2\int \frac{1}{\sin^2\theta(1-\sin^2\theta)}\cos\theta\,d\theta \\
&= 2\int \frac{1}{u^2(1-u^2)}\,du.
\end{aligned}$$

At this point, we have a rational function, and we can integrate all rational functions with partial fraction decomposition. Following those steps, we write that

$$\frac{2}{u^2(1-u^2)} = \frac{2}{u^2(1+u)(1-u)} = \frac{Au+B}{u^2} + \frac{C}{1+u} + \frac{D}{1-u}.$$

Clearing denominators, we find that

$$2 = (Au+B)(1+u)(1-u) + Cu^2(1-u) + Du^2(1+u).$$

Plugging in $u = 1$ so that most of the terms cancel, and we find that $2 = D(1)(2)$, so $D = 1$. Plugging in $u = -1$ so that most of the terms cancel, we find that $2 = C(-1)^2(2)$, so $C = 1$. Plugging in $u = 0$ so that most of the terms cancel, we find that $2 = B(1)(1)$, so $B = 2$. Finally, plugging in $u = 2$, we find that

$$\begin{aligned} 2 &= (2A+B)(3)(-1) + C(4)(-1) + D(4)(3) \\ &= -6A - 3B - 4C + 12D \\ &= -6A - 6 - 4 + 12 \\ &= -6A + 2. \end{aligned}$$

Thus, $A = 0$. Therefore,

$$\begin{aligned} \int \frac{2}{u^2(1-u^2)}\,du &= \int \left(\frac{2}{u^2} + \frac{1}{1+u} + \frac{1}{1-u}\right) \\ &= -\frac{2}{u} + \ln|1+u| - \ln|1-u| + C \\ &= -\frac{2}{u} + \ln\left|\frac{1+u}{1-u}\right| + C \end{aligned}$$

From the triangle, we see that $u = \sin\theta = \frac{\sqrt{x^2-1}}{x}$, so

$$\begin{aligned} -\frac{2}{u} + \ln\left|\frac{1+u}{1-u}\right| + C &= -\frac{2x}{\sqrt{x^2-1}} + \ln\left|\frac{1+\frac{\sqrt{x^2-1}}{x}}{1-\frac{\sqrt{x^2-1}}{x}}\right| + C \\ &= \boxed{-\frac{2x}{\sqrt{x^2-1}} + \ln\left(\frac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}}\right) + C.} \end{aligned}$$

An alternate form of the answer is

$$\boxed{-\frac{2x}{\sqrt{x^2-1}} + 2\ln\left(x+\sqrt{x^2-1}\right) + C,}$$

using the fact that

$$\left(x+\sqrt{x^2-1}\right)\left(x-\sqrt{x^2-1}\right) = x^2 - (x^2-1) = 1.$$

7.8: Improper Integrals. For each the following integrals, evaluate the integral if it converges. If it diverges, write "diverges because" and write down the limit that does not exist.

(1) $\displaystyle\int_0^\infty \frac{1}{(x-3)^2}\,dx.$

Solution: We first compute the antiderivative.

$$\int \frac{1}{(x-3)^2}\,dx = \int \frac{1}{(x-3)^2}\,d(x-3) = -\frac{1}{x-3} + C.$$

We identify the bad values. One of the bounds is ∞, so that's bad. In addition, the expression $\frac{1}{(x-3)^2}$ goes off to infinity when $x = 3$. We split up our domain into parts so that each part has one bad value at one of the bounds. We have that

$$\int_0^\infty \frac{1}{(x-3)^2}\,dx = \int_0^3 \frac{1}{(x-3)^2}\,dx + \int_3^4 \frac{1}{(x-3)^2}\,dx + \int_4^\infty \frac{1}{(x-3)^2}\,dx.$$

Evaluating the first piece, we replace the bad bound with a limit.

$$\begin{aligned}\int_0^3 \frac{1}{(x-3)^2}\,dx &= \lim_{t\to 3}\int_0^t \frac{1}{(x-3)^2}\,dx \\ &= \lim_{t\to 3}\left(-\frac{1}{x-3}\right)\Big|_0^t \\ &= \boxed{\lim_{t\to 3}\left(-\frac{1}{t-3}+\frac{1}{3}\right).}\end{aligned}$$

This limit does not exist, so $\int_0^3 \frac{1}{(x-3)^2}\,dx$ diverges, and so $\int_0^\infty \frac{1}{(x-3)^2}\,dx$ $\boxed{\text{diverges}}$ as well.

We could also have found that the second piece is divergent because the limit

$$\boxed{\lim_{t\to 3}\left(-1+\frac{1}{t-3}\right)}$$

does not exist.

(2) $\displaystyle\int_0^\infty \cos(3x)\,dx.$

Solution: The antiderivative is

$$\int \cos(3x)\,dx = \tfrac{1}{3}\sin(3x)+C.$$

The expression $\cos(3x)$ never goes off to infinity, but we do have a bound of ∞, so that's our only bad bound. We replace it with a limit.

$$\begin{aligned}\int_0^\infty \cos(3x)\,dx &= \lim_{t\to\infty}\int_0^t \cos(3x)\,dx \\ &= \lim_{t\to\infty}\left(\tfrac{1}{3}\sin(3x)\right)\Big|_0^t \\ &= \boxed{\lim_{t\to\infty}\tfrac{1}{3}\sin(3t).}\end{aligned}$$

This limit does not exist. As $t\to\infty$, the function $\sin(3t)$ bounces back and forth between -1 and 1 without approaching any one value.

(3) $\displaystyle\int_2^3 \frac{1}{\sqrt{3-x}}\,dx.$

Solution: We compute the antiderivative.

$$\int \frac{1}{\sqrt{3-x}}\,dx = \int -\frac{1}{\sqrt{3-x}}\,d(3-x) = -2\sqrt{3-x}+C.$$

The expression $\frac{1}{\sqrt{3-x}}$ goes off to infinity when $x=3$. That's our bad point, so our integral already has exactly one bad point at one of the bounds. Replacing the bad bound with a limit, we find that

$$\begin{aligned}\int_2^3 \frac{1}{\sqrt{3-x}}\,dx &= \lim_{t\to 3}\int_2^t \frac{1}{\sqrt{3-x}}\,dx \\ &= \lim_{t\to 3}\left(-2\sqrt{3-x}\right)\Big|_2^t \\ &= \lim_{t\to 3}\left(-2\sqrt{3-t}+2\sqrt{3-2}\right) \\ &= -2\cdot 0+2\sqrt{1} = \boxed{2}.\end{aligned}$$

(4) $\displaystyle\int_e^\infty \frac{1}{x(\ln x)^3}\,dx.$

Solution: We can compute the antiderivative using the substitution $u = \ln x$, $du = \frac{1}{x}\,dx$.

$$\int \frac{1}{x(\ln x)^3}\,dx = \int \frac{1}{u^3}\,du = -\frac{1}{2}\cdot\frac{1}{u^2} + C = -\frac{1}{2(\ln x)^2} + C.$$

To find the bad x values, we see that we have one at ∞ from the bounds. The expression $\frac{1}{x(\ln x)^3}$ goes off to infinity either when $x = 0$ or when $\ln x = 0$, which happens when $x = 1$. Neither of these is in the range $e \le x < \infty$, so there's no need to split up the integral. Indeed, $\int_e^\infty \frac{1}{x(\ln x)^3}$ has exactly one bad point, namely ∞, and it is at one of the bounds. We replace the bad bound with a limit and compute.

$$\begin{aligned}
\int_e^\infty \frac{1}{x(\ln x)^3}\,dx &= \lim_{t\to\infty}\int_e^t \frac{1}{x(\ln x)^3}\,dx \\
&= \lim_{t\to\infty}\left(-\frac{1}{2(\ln x)^2}\right)\Bigg|_e^t \\
&= \lim_{t\to\infty}\left(-\frac{1}{2(\ln t)^2} + \frac{1}{2(\ln e)^2}\right) \\
&= 0 + \tfrac{1}{2} = \boxed{\tfrac{1}{2}}.
\end{aligned}$$

To compute the limit, we know that, as $t \to \infty$, we have $\ln t \to \infty$, and so $\frac{1}{(\ln t)^2} \to 0$.

8.1: Arclength. Find the arclength of the following curves.

(1) $y = 1 + 2x^{\frac{3}{2}}$ from $x = 0$ to $x = 1$.

Solution: We compute

$$\frac{dy}{dx} = 3x^{1/2}$$

Thus, the arclength is

$$\begin{aligned}
\int_0^1 \sqrt{1 + \left(3x^{1/2}\right)^2}\,dx &= \int_0^1 \sqrt{1+9x}\,dx \\
&= \int_0^1 \sqrt{1+9x}\cdot\tfrac{1}{9}\,d(1+9x) \\
&= \tfrac{2}{3}(1+9x)^{3/2}\cdot\tfrac{1}{9}\Big|_0^1 \\
&= \boxed{\tfrac{2}{27}\left(10^{3/2}-1\right)} = \boxed{\tfrac{2}{27}\left(10\sqrt{10}-1\right).}
\end{aligned}$$

(2) $x = \frac{1}{3}\sqrt{y}(y-3)$ $1 \le y \le 16$

Solution: We compute

$$\begin{aligned}
\frac{dx}{dy} &= \tfrac{1}{3}\left(\tfrac{1}{2}y^{-1/2}(y-3) + y^{1/2}\right) \\
&= \tfrac{1}{2}y^{1/2} - \tfrac{1}{2}y^{-1/2}.
\end{aligned}$$

Then, the arclength is

$$\begin{aligned}\int_1^{16} \sqrt{1+\left(\tfrac{1}{2}y^{1/2}-\tfrac{1}{2}y^{-1/2}\right)^2}\,dy &= \int_1^{16} \sqrt{1+\tfrac{1}{4}y-\tfrac{1}{2}+\tfrac{1}{4}y^{-1}}\,dy \\ &= \int_1^{16} \sqrt{\tfrac{1}{4}y+\tfrac{1}{2}+\tfrac{1}{4}y^{-1}}\,dy \\ &= \int_1^{16} \sqrt{\left(\tfrac{1}{2}y^{1/2}+\tfrac{1}{2}y^{-1/2}\right)^2} \\ &= \int_1^{16} \left(\tfrac{1}{2}y^{1/2}+\tfrac{1}{2}y^{-1/2}\right) \\ &= \left(\tfrac{1}{3}y^{3/2}+y^{1/2}\right)\Big|_1^{16} \\ &= \left(\tfrac{64}{3}+4\right)-\left(\tfrac{1}{3}+1\right) = \boxed{24}.\end{aligned}$$

(3) $y = \sqrt{x-x^2} + \arcsin(\sqrt{x})$ from $x = 0$ to $x = \frac{1}{4}$

Solution: We begin by computing $\frac{dy}{dx}$. To do that, we need to look up or use the inverse function theorem to find that the derivative of $\arcsin(x)$ is $\frac{1}{\sqrt{1-x^2}}$. We find that

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2\sqrt{x-x^2}}\cdot(1-2x) + \frac{1}{\sqrt{1-\sqrt{x}^2}}\cdot\frac{1}{2\sqrt{x}} \\ &= \frac{1-2x}{2\sqrt{x-x^2}} + \frac{1}{2\sqrt{(1-x)x}} \\ &= \frac{1-2x}{2\sqrt{x-x^2}} + \frac{1}{2\sqrt{x-x^2}} \\ &= \frac{2-2x}{2\sqrt{x-x^2}} \\ &= \frac{2(1-x)}{2\sqrt{x(1-x)}} \\ &= \frac{\sqrt{1-x}}{\sqrt{x}}.\end{aligned}$$

Now, we compute that the arclength is

$$\begin{aligned}\int_0^{1/4} \sqrt{1+\left(\frac{\sqrt{1-x}}{\sqrt{x}}\right)^2}\,dx &= \int_0^{1/4} \sqrt{1+\frac{1-x}{x}}\,dx \\ &= \int_0^{1/4} \sqrt{\frac{x+1-x}{x}}\,dx \\ &= \int_0^{1/4} \sqrt{\frac{1}{x}}\,dx \\ &= \int_0^{1/4} x^{-1/2}\,dx \\ &= 2\sqrt{x}\Big|_0^{1/4} = \boxed{1}.\end{aligned}$$

(4) $x^2 = (y-4)^3$ from $(1,5)$ to $(8,8)$.

Solution: It's easiest to solve for x. Since $x > 0$ in the range we're interested in, we find that $x = (y-4)^{3/2}$. Then, we compute

$$\frac{dx}{dy} = \tfrac{3}{2}(y-4)^{1/2}.$$

Next, we compute the arclength, keeping in mind that we're writing everything interms of y, so the relevant bounds are $5 \leq y \leq 8$. We compute

$$\begin{aligned}
\int_5^8 \sqrt{1+\left(\tfrac{3}{2}(y-4)^{1/2}\right)^2}\,dy &= \int_5^8 \sqrt{1+\tfrac{9}{4}(y-4)} \\
&= \int_5^8 \sqrt{\tfrac{9}{4}y-8}\,dy \\
&= \int_5^8 \sqrt{\tfrac{9}{4}y-8}\cdot\tfrac{4}{9}\,d(\tfrac{9}{4}-8) \\
&= \tfrac{2}{3}\left(\tfrac{9}{4}y-8\right)^{3/2}\cdot\tfrac{4}{9}\Big|_5^8 \\
&= \boxed{\tfrac{8}{27}\left(10^{3/2}-\left(\tfrac{13}{4}\right)^{3/2}\right)} \\
&= \boxed{\frac{80\sqrt{10}}{27}-\frac{13\sqrt{13}}{27}.}
\end{aligned}$$