Math 132 - October 30, 2017
Solutions

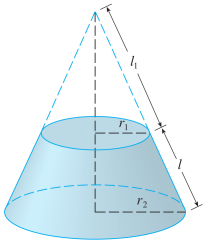
8.1 Arc Length

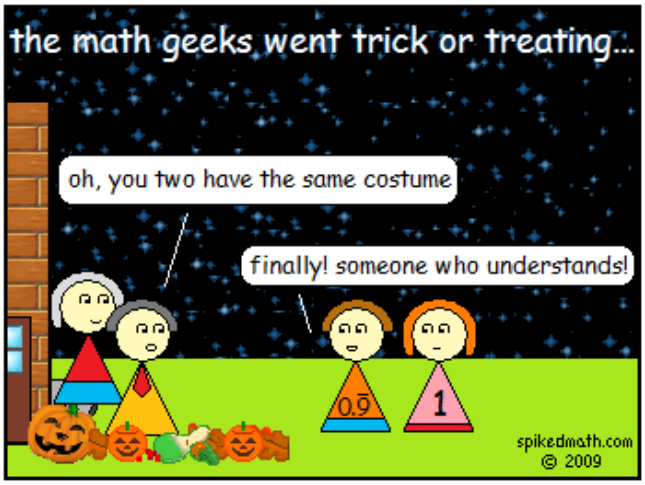
$$L = \int ds = \int_a^b \sqrt{1 + (f'(x))^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

8.2 Surface Area

$$S = \int_a^b dS = \int_a^b 2\pi r\, ds = \int_a^b 2\pi r \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

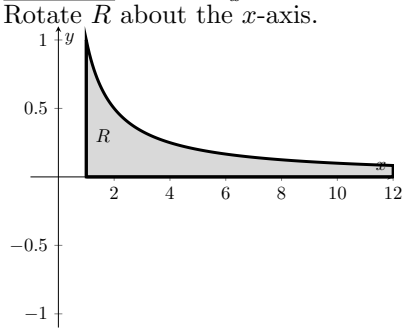
$$\text{Area(Frustum)} = 2\pi l \left(\frac{r_1 + r_2}{2}\right) = 2\pi\, r_{\text{ave}}\, l$$

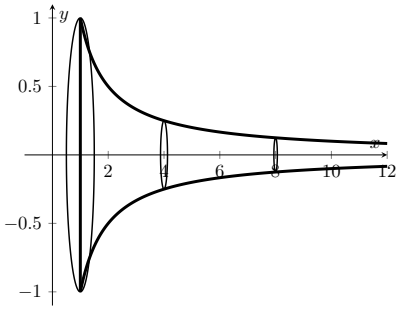




Warm-up Problems

1. **Clicker** Let $y = \frac{1}{x}$ on the interval $[1, \infty)$. Let R be the region between the graph and the x-axis about the x-axis. Rotate R about the x-axis.





A = The area of R
V = The volume of the generated solid
S = The surface area of the generated surface

Select the true statements about A, S and V.

Selection	A	V	S
A.	∞	∞	∞
B. **Correct**	∞	Finite	∞
C.	∞	∞	Finite
D.	∞	Finite	Finite
E.	Finite	Finite	Finite

Solution: This is a fun problem. We haven't really worked with surface area so that one will be new (see below).

$$A = \int_1^\infty \frac{1}{x}\, dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\, dx = \lim_{t\to\infty} \ln x \Big|_1^t = \lim_{t\to\infty} \ln t = \infty$$

For volume, we use washers.

$$V = \int_1^\infty \pi \cdot \frac{1}{x^2}\, dx = \lim_{t\to\infty} \pi \int_1^t \frac{1}{x^2}\, dx = \lim_{t\to\infty} -\frac{\pi}{x}\Big|_1^t = \lim_{t\to\infty} -\frac{\pi}{t} + \pi = \pi$$

Length (not asked for, but here it is, obviously ∞ by looking at the picture, but ...)

$$L = \int_1^\infty \sqrt{1 + \frac{1}{x^4}}\, dx = \lim_{t\to\infty} \int_1^t \frac{\sqrt{x^4+1}}{x^2}\, dx$$

This integral, apparently can be computed (see Wolfram Alpha) but that looks too difficult to me. Instead we can compare. Note that

$$\sqrt{1 + \frac{1}{x^4}} > \sqrt{1+0} = 1$$

Thus

$$L = \int_1^\infty \sqrt{1 + \frac{1}{x^4}}\, dx = \lim_{t\to\infty} \int_1^t \sqrt{1 + \frac{1}{x^4}}\, dx > \lim_{t\to\infty} \int_1^t 1\, dx = \infty$$

and thus the length must be infinity.

For surface area....

$$SA = \int_1^\infty dS = \int_1^\infty 2\pi r\, ds = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\, dx = 2\pi \int_1^\infty \frac{\sqrt{x^4+1}}{x^3}\, dx$$

Again, not one we want to compute but we can compare

$$\frac{\sqrt{x^4+1}}{x^3} > \frac{\sqrt{x^4+0}}{x^3} = \frac{1}{x}$$

Thus

$$SA = 2\pi \int_1^\infty \frac{\sqrt{x^4+1}}{x^3}\, dx > 2\pi \int_1^\infty \frac{1}{x}\, dx = \infty$$

Thus, we have finite volume, but infinite surface area.

2. Set up arc length integral (compute if you have time!).

   (a) Length of $y = x^3$ from $x = 1$ to $x = 4$.
   Solution:

   $$L = \int_1^4 \sqrt{1 + (3x^2)^2}\, dx \approx 63.124$$

   (Apparently there are some techniques to integrate this but I'm not sure what I would do on this one. In any case, Wolfram Alpha has an explicit answer.)

   (b) Length of $x = y^3$ from $y = 1$ to $y = 4$.
   Solution: The point here is that x and y have changed roles

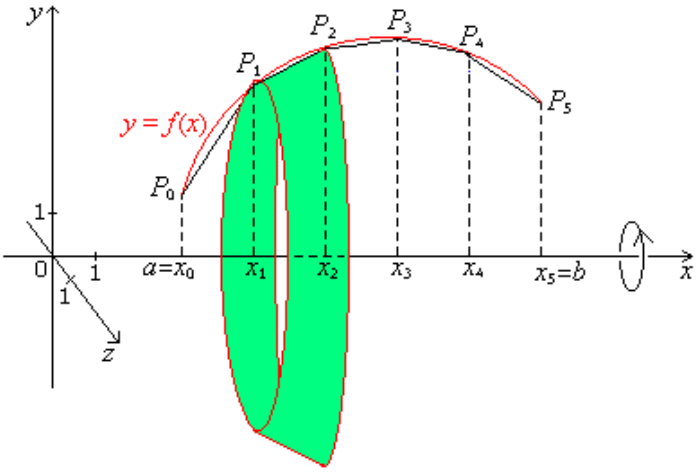
   $$L = \int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_1^4 \sqrt{1 + (3y^2)^2}\, dy \approx 63.124$$

   (c) Length of $y = \frac{1}{6}x^3 + \frac{1}{2x}$ from $x = 1$ to $x = 2$
   Solution:

   $$L = \int_1^2 \sqrt{1 + \left(\frac{1}{2}x^2 - \frac{1}{2x^2}\right)^2}\, dx = \int_1^2 \frac{1}{2}x^2 + \frac{1}{2x^2}\, dx = \frac{17}{12}$$

Class Problems

Lecture Notes: **Surface Area.** We are going to work on surface area of solids of revolution. For more general areas, look ahead to the material in any calc 3 textbook.
The idea is to take a surface of revolution, chop it up, which yeilds small frustum-like rings and add up these areas of frustums.



When you do this, you get

$$\begin{aligned} S &\approx \sum (\text{Area Frustum}) \\ &= \sum 2\pi r l \end{aligned}$$

Taking limits and integrating gives

$$S = \int_a^b 2\pi r \, ds = \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\, dx$$

Here's an example.
Find the area of a parabola $y = x^2$ rotated about the y-axis from $x = 0$ to $x = 1$. Draw the picture. You can do this integral either dx or dy

$$\begin{aligned} S &= \int_0^1 2\pi r l \, ds \\ &= \int_0^1 2\pi x\sqrt{1+(dy/dx)^2}\, dx \\ &= \int_0^1 2\pi x\sqrt{1+4x^2}\, dx \\ &= \frac{1}{6}\pi(1+4x^2)^{3/2}\Big|_0^1 = \frac{1}{6}\pi(5^{3/2}-1) \approx 5.3304 \end{aligned}$$

Here's the dy way:

$$\begin{aligned} S &= \int_0^1 2\pi r l \, ds \\ &= \int_0^1 2\pi x\sqrt{1+(dx/dy)^2}\, dy \\ &= \int_0^1 2\pi\sqrt{y}\sqrt{1+\frac{1}{4y}}\, dy \\ &= \int_0^1 \pi\sqrt{1+4y}\, dy = \frac{1}{6}\pi(5^{3/2}-1) \end{aligned}$$

3. Find the area of $y = \sqrt{x}$ on the interval $[3/4, 15/4]$ rotated about the x-axis. (Just set up, compute if you have time.)

Solution:

$$S = \int_{3/4}^{15/4} 2\pi\sqrt{x}\sqrt{1+\frac{1}{4x}}\,dx = \frac{28}{3}$$

4. Find the area of $x = \frac{y^3}{3}$ on the interval $y = 0$ to $y = 1$ rotated about the y-axis. (Just set up, compute if you have time.)

   Solution:

$$S = \int_0^1 2\pi\frac{y^3}{3}\sqrt{1+y^4}\,dx = \frac{\pi}{9}(2\sqrt{2}-1)$$

5. Find the area of $y = \sin x$ from $x = 0$ to $x = \pi$, rotated about the x-axis. (Just set up, compute if you have time.)

   Solution:

$$S = \int_0^\pi 2\pi\sin x\sqrt{1+\cos^2 x}\,dx \approx 2.295587\ldots$$

6. Find the surface area of the arc $y = 4x + 1$, $1 \le x \le 3$, rotated about the x-axis.
   Solution:

$$S = \int_1^3 2\pi r\,ds = \int_1^3 2\pi r\,\sqrt{1+4^2}\,dx = \int_1^3 2\pi(4x+1)\,\sqrt{1+4^2}\,dx = 36\pi\sqrt{17}$$

7. Find the surface area of the arc $y = 4x + 1$, $1 \le x \le 3$, rotated about the y-axis.
   Solution: Solve for $x = (y-1)/4$ from $y = 5$ to $y = 13$.

$$S = \int_5^{13} 2\pi r\,ds = \int_5^{13} 2\pi x\,\sqrt{1+(1/16)}\,dy = \int_5^{13} 2\pi(1/4)(y-1)\,\sqrt{1+(1/16)}\,dy = 8\pi\sqrt{17}$$

   OR

$$S = \int_1^3 2\pi r\,ds = \int_1^3 2\pi x\,\sqrt{1+16}\,dx = 8\pi\sqrt{17}$$

8. Find the surface area of the arc $y = x^2$, $1 \le x \le 4$, rotated about the x-axis.
   Solution:

$$S = \int_1^4 2\pi r\,ds = \int_1^4 2\pi y\,\sqrt{1+(2x)^2}\,dx = \int_1^4 2\pi(x^2)\,\sqrt{1+4x^2}\,dx \approx 812.7563\ldots$$

9. Find the surface area of the arc $y = x^2$, $1 \le x \le 4$, rotated about the axis $y = -3$.
   Solution:

$$S = \int_1^4 2\pi r\,ds = \int_1^4 2\pi(y-(-3))\,\sqrt{1+(2x)^2}\,dx = \int_1^4 2\pi(x^2+3)\,\sqrt{1+4x^2}\,dx \approx 1101.90268\ldots$$

10. Find the surface area of the arc $y = x^2$, $1 \le x \le 4$, rotated about the y-axis.
    Solution: Note this is actually $x = \sqrt{y}$ from $1 \le y \le 16$.

$$S = \int_1^{16} 2\pi r\,ds = \int_1^{16} 2\pi x\,\sqrt{1+\frac{1}{4y}}\,dy = \int_1^{16} 2\pi(\sqrt{y})\,\sqrt{1+\frac{1}{4y}}\,dy \approx 268.536226\ldots$$

    OR

$$S = \int_1^4 2\pi r\,ds = \int_1^4 2\pi x\,\sqrt{1+4x^2}\,dx \approx 268.536226\ldots$$

11. Find the surface area of the arc $y = x^2$, $1 \le x \le 4$, rotated about the axis $x = -5$.
    Solution: Note this is actually $x = \sqrt{y}$ from $1 \le y \le 16$.

$$S = \int_1^{16} 2\pi r\,ds = \int_1^{16} 2\pi(x+5)\,\sqrt{1+\frac{1}{4y}}\,dy = \int_1^{16} 2\pi(\sqrt{y}+5)\,\sqrt{1+\frac{1}{4y}}\,dy \approx 750.44682269\ldots$$

    OR

$$S = \int_1^4 2\pi r\,ds = \int_1^4 2\pi(x+5)\,\sqrt{1+4x^2}\,dy \approx 750.44682269\ldots$$

12. Find the surface area of the arc $y = x^2$, $1 \le x \le 4$, rotated about the axis $x = 5$.
Solution: Note this is actually $x = \sqrt{y}$ from $1 \le y \le 16$.

$$S = \int_1^{16} 2\pi r \, ds = \int_1^{16} 2\pi(5 - x) \sqrt{1 + \frac{1}{4y}} \, dy = \int_1^{16} 2\pi(5 - \sqrt{y}) \sqrt{1 + \frac{1}{4y}} \, dy \approx 213.37437\ldots$$

OR

$$S = \int_1^{4} 2\pi r \, ds = \int_1^{4} 2\pi(5 - x) \sqrt{1 + 4x^2} \, dy \approx 213.37437\ldots$$