Math 132 - October 18, 2017
Solutions

7.4 Partial Fractions

Partial Fraction Steps:

1. Long Division (if deg top > deg bottom)
2. Factor the denominator (linear and quadratic irreducibles)
3. Determine the "form" (linear, quadratic, repeated)
4. Solve for constants
5. Integrate term by term

Warm-up Problems

1. **Clicker** Solve the system of equations

$$\begin{aligned} 2A + 2B + C &= 13 \\ 3A - B + 2C &= -6 \\ -2A &= 1 \end{aligned}$$

$B =$??

(a) $-1/2$

(b) $13/2$ **Correct**

(c) 1

Solution: Solve this linear system of equations. I generally use row reduction and matrices, but you can use elimination (my 2nd choice) or substitution (my last choice, usually gets too messy).

$$\begin{bmatrix} 2 & 2 & 1 & 13 \\ 3 & -1 & 2 & -6 \\ -2 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 13/2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Alternatively, you can easily solve for $A = -1/2$, plug that in to the equations to get

$$\begin{aligned} 2B + C &= 14 \\ -B + 2C &= -9/2 \end{aligned}$$

Multiple the top equation by -2 and add equations:

$$\begin{aligned} -4B - 2C &= -28 \\ -B + 2C &= -9/2 \end{aligned}$$

to get the equation $-5B = -65/2$ and $B = 13/2$.

2. Find the general integrals below. (These are the types of integrals that should appear when partial fractions is conducted correctly.)

(a) $\displaystyle\int \frac{1}{ax+b}\,dx$

Solution: Use substitution, $u = ax + b$:

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln(ax+b) + C$$

(b) $\displaystyle\int \frac{1}{(ax+b)^2}\,dx$

Solution: Use substitution $u = ax + b$:

$$\int \frac{1}{(ax+b)^2}\,dx = -\frac{1}{a(ax+b)} + C$$

Note: This is similar to any integral of the form $\frac{1}{(ax+b)^n}$ for any $n > 1$.

(c) $\displaystyle\int \frac{x}{x^2+a^2}\,dx$

Solution: Use substitution $u = x^2 + a^2$:

$$\int \frac{x}{x^2+a^2}\,dx = \frac{1}{2}\ln(x^2+a^2) + C$$

(d) $\displaystyle\int \frac{x}{(x^2+a^2)^2}\,dx$

Solution: Use substitution $u = x^2 + a^2$:

$$\int \frac{x}{(x^2+a^2)^2}\,dx = -\frac{1}{2(x^2+a^2)} + C$$

Note: This is similar to any integral of the form $\frac{x}{(x^2+a)^n}$ for any $n > 1$.

(e) $\displaystyle\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$

Solution: You can use substitution $x = au$:

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

(f) $\displaystyle\int \frac{1}{(x^2+a^2)^2}\,dx$

Solution: Use Trig subsitution $x = a\tan\theta$:

$$\int \frac{1}{(x^2+a^2)^2}\,dx = \frac{1}{2a^3}\arctan\left(\frac{x}{a}\right) + \frac{x}{2a^2(x^2+a^2)} + C$$

In this case, integrals of the form $\int \frac{1}{(x^2+a^2)^n}\,dx$ are clearly trickier. You can do these using trig substitution ($x = a\tan\theta$)

$$\begin{aligned}\int \frac{1}{(x^2+a^2)^n}\,dx &= \int \frac{1}{(a^2\tan^2\theta + a^2)^n}\,d\theta \qquad x = a\tan\theta \\ &= \frac{1}{a^{2n}}\int \frac{1}{(\tan^2\theta+1)^n}\,d\theta = \frac{1}{a^{2n}}\int \frac{1}{\sec^{2n}\theta}\,d\theta = \frac{1}{a^{2n}}\int \cos^{2n}\theta\,d\theta\end{aligned}$$

Class Problems

Lecture Notes: Here's how partial fractions works, with an example. We want to integrate $\displaystyle\int \frac{4x^4+1}{2x^3+3x^2-2x}\,dx$

$$\begin{aligned}\frac{4x^4+1}{2x^3+3x^2-2x} &= 2x - 3 + \frac{13x^2-6x+1}{2x^3+3x^2-2x} && \text{(Long Division)} \\ \frac{13x^2-6x+1}{2x^3+3x^2-2x} &= \frac{13x^2-6x+1}{x(x+2)(2x-1)} && \text{(Factor denominator)} \\ &= \frac{A}{x} + \frac{B}{x+2} + \frac{C}{2x-1} && \text{(Determine correct form of P.F. Decomposition)}\end{aligned}$$

Now you solve for A, B, C. Several tricks help do this quicker, but the general method is:

$$\begin{aligned}&\frac{13x^2-6x+1}{2x^3+3x^2-2x} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{2x-1} \\ &\left(\frac{13x^2-6x+1}{2x^3+3x^2-2x} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{2x-1}\right)(2x^3+3x^2-2x) \\ &13x^2-6x+1 = A(x+2)(2x-1) + B(x)(2x-1) + C(x)(x+2) \\ &13x^2-6x+1 = (2A+2B+C)x^2 + (3A-B+2C)x + (-2A)\end{aligned}$$

This gives three equations and three unknowns:

$$\begin{aligned} 2A + 2B + C &= 13 \\ 3A - B + 2C &= -6 \\ -2A &= 1 \end{aligned}$$

Solve this linear system of equations. I generally use row reduction and matrices, but you can use elimination (my 2nd choice) or substitution (my last choice, usually gets too messy).

$$\begin{bmatrix} 2 & 2 & 1 & 13 \\ 3 & -1 & 2 & -6 \\ -2 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 13/2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Thus we have $A = -\frac{1}{2}$, $B = \frac{13}{2}$, and $C = 1$ giving

$$\frac{13x^2 - 6x + 1}{2x^3 + 3x^2 - 2x} = -\frac{1}{2x} + \frac{13}{2(x+2)} + \frac{1}{2x-1}$$

and thus

$$\frac{4x^4 + 1}{2x^3 + 3x^2 - 2x} = 2x - 3 - \frac{1}{2x} + \frac{13}{2(x+2)} + \frac{1}{2x-1}$$

We can now integrate:

$$\begin{aligned} \int \frac{4x^4 + 1}{2x^3 + 3x^2 - 2x}\, dx &= \int \left(2x - 3 - \frac{1}{2x} + \frac{13}{2(x+2)} + \frac{1}{2x-1}\right)\, dx \\ &= \int (2x - 3)\, dx - \int \frac{1}{2x}\, dx + \int \frac{13}{2(x+2)}\, dx + \int \frac{1}{2x-1}\, dx \\ &= x^2 - 3x - \frac{1}{2}\ln|x| + \frac{13}{2}\ln|x+2| + \frac{1}{2}\ln|2x-1| + C \end{aligned}$$

Lecture Notes: Key points

- Long division should be straightforward (perhaps not fun, but should be straghtforward)
- Factoring should be straightforward, but sometimes difficult.
- The correct form takes some getting used to. The tricks are:
  - Linear factors get numerators of A, B, C, etc:

$$\frac{2x^3 + 3}{(x+1)(x+2)(x+3)(x+4)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3} + \frac{D}{x+4}$$

  - Quadratic factors get numerators of $Ax + B$, $Cx + D$, etc:

$$\frac{2x^3 + 3}{(x^2+1)(x^2+2)(x^2+3)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2} + \frac{Ex+F}{x^2+3}$$

  - Repeat factors get repeated denominators:

$$\frac{2x^3 + 3}{(x+1)(x+2)^3(x^2+3)^2} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2} + \frac{D}{(x+2)^3} + \frac{Ex+F}{x^2+3} + \frac{Gx+H}{(x^2+3)^2}$$

  - There should be no factors of degree 3 and higher.
- Solving the system of equations is often a big pain but it will always be linear, which means it is straightforward. There are online solvers that should work. If you have to solve a big system by hand, you will almost certainly want to use matrices and row reduction (easy to learn or just take Math 309).

- Once you have it solved, the integrals should all be easy and all of the form in Problem 2.
- You can/should check everything with an online "partial fractions calculator".

3. **Clicker** Determine the correct partial fraction form. How many constants are needed? $\int \frac{1}{(x-1)(x-2)}\,dx$

   (a) 1 (b) 2 **Correct** (c) 3 (d) 4 (e) 5

   Solution: $\frac{A}{x-1}+\frac{B}{x-2}$

$$\int \frac{1}{(x-1)(x-2)}\,dx = \int\left(\frac{-1}{x-1}+\frac{1}{(x-2)}\right)dx$$
$$= -\ln|x-1|+\ln|x-2|+C$$

4. **Clicker** Determine the correct partial fraction form. How many constants are needed? $\int \frac{1}{(x-1)(x-2)^2}\,dx$

   (a) 1 (b) 2 (c) 3 **Correct** (d) 4 (e) 5

   Solution: $\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{(x-3)^2}$

$$\int \frac{1}{(x-1)(x-2)^2}\,dx = \int\left(\frac{1}{x-1}+\frac{-1}{x-2}+\frac{1}{(x-2)^2}\right)dx$$
$$= \ln|x-1|-\ln|x-2|-\frac{1}{x-2}+C$$

5. **Clicker** Determine the correct partial fraction form. How many constants are needed? $\int \frac{1}{(x-1)(x^2+4)}\,dx$

   (a) 1 (b) 2 (c) 3 (d) 4 **Correct** (e) 5

   Solution: $\frac{A}{x-1}+\frac{Bx+C}{x^2+4}$

$$\int \frac{1}{(x-1)(x^2+4)}\,dx = \int\left(\frac{1/5}{x-1}+\frac{(-1/5)x-(1/5)}{x^2+4}\right)dx$$
$$= \frac{1}{5}\ln|x-1|-\frac{1}{10}\ln(x^2+4)-\frac{1}{10}\arctan(x/2)+C$$

6. **Clicker** Determine the correct partial fraction form. How many constants are needed? $\int \frac{2}{(x-1)(x^2+4)^2}\,dx$

   (a) 1 (b) 2 (c) 3 (d) 4 (e) 5 **Correct**

   Solution: $\frac{A}{x-1}+\frac{Bx+C}{x^2+4}+\frac{Dx+E}{(x^2+4)^2}$

$$\int \frac{2}{(x-1)(x^2+4)^2}\,dx = \int\left(\frac{1/25}{x-1}+\frac{(-1/25)x-(1/25)}{x^2+4}+\frac{(-1/5)x+(-1/5)}{(x^2+4)^2}\right)dx$$
$$= \frac{1}{25}\ln|x-1|-\frac{1}{50}\ln(x^2+4)-\frac{1}{50}\arctan(x/2)-\frac{1}{80}\arctan(x/2)+\frac{x-4}{20(x^2+4)}+C$$

7. Determine the correct form for the following

   (a) $\int \frac{x+3}{x(3x+2)}\,dx = \int\left(\frac{3}{2x}-\frac{7}{2(3x+2)}\right)dx = \frac{3}{2}\ln|x|-\frac{7}{6}\ln|3x+2|+C$

   Solution: $=\frac{A}{x}+\frac{B}{3x+2}$

   (b) $\int \frac{x+3}{x(3x+2)^3}\,dx = \int\left(\frac{3}{8x}-\frac{9}{8(3x+2)}-\frac{9}{4(3x+2)^2}-\frac{7}{2(3x+2)^3}\right)dx$
   $= \frac{3}{8}\ln|x|-\frac{3}{8}\ln|3x+2|+\frac{3}{4(3x+2)}+\frac{7}{12(3x+2)^2}+C$

   Solution: $\frac{A}{x}+\frac{B}{3x+2}+\frac{C}{(3x+2)^2}+\frac{D}{(3x+2)^3}$

(c) $\displaystyle\int \frac{x+3}{(3x+2)^3}\,dx = \int\left(\frac{1}{3(3x+2)^2}+\frac{7}{3(3x+2)^3}\right)\,dx = -\frac{1}{9(3x+2)}-\frac{7}{18(3x+2)^2}+C$

Solution: $\frac{A}{3x+2}+\frac{B}{(3x+2)^2}+\frac{C}{(3x+2)^3}$

(d) $\displaystyle\int \frac{3}{(3x+2)^3}\,dx = -\frac{1}{2(3x+2)^2}+C$

Solution: (Trick Question) $\frac{A}{(3x+2)^2}$

(e) $\displaystyle\int \frac{x^2+x+1}{x(x^2+9)}\,dx = \int\left(\frac{1}{9x}+\frac{8x+9}{9(x^2+9)}\right)\,dx = \frac{1}{9}\ln|x|+\frac{4}{9}\ln|x^2+9|+\frac{1}{3}\arctan(x/3)+C$

Solution: $\frac{A}{x}+\frac{Bx+C}{x^2+9}$

(f) $\displaystyle\int \frac{x^2+x+1}{x(x^2+9)(x^2+4)}\,dx = \int\left(\frac{1}{36x}-\frac{8x+9}{45(x^2+9)}+\frac{3x+4}{20(x^2+4)}\right)\,dx$
$= \displaystyle\frac{1}{36}\ln|x|-\frac{4}{45}\ln(x^2+9)-\frac{1}{15}\arctan(x/3)+\frac{3}{4}\ln|x^2+4|+\frac{1}{10}\arctan(x/2)+C$

Solution: $\frac{A}{x}+\frac{Bx+C}{x^2+9}+\frac{Dx+E}{x^2+4}$

8. For the previous problems, solve for the constants and compute the integrals.