Math 132 - October 6, 2017
Solutions

Sections Covered: 7.2

Trig Identies:

- $1 + \tan^2\theta = \sec^2\theta$
- $\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$
- $\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$
- $\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$
- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

Trig Integral Stategies

$\int \sin^m x \ \cos^n x \ dx$

- If m, n (or both) are odd. Use either $u = \sin x$ or $u = \cos x$.
- If m, n are both even, use half angle identities.

$\int \tan^m x \ \sec^n x \ dx$

- Use either $u = \tan x$ or $u = \sec x$
- If the above doesn't work, try something else.

$\int \sin(nx) \ \cos(mx) \ dx$

- Use product to sum formulas

$\int \sec x \ dx$ or $\int \csc x \ dx$

- This is a trick. Just be ready to look this integral up if needed.

Warm-up Problems

1. Integrate with $u = \sin x$: $\int \sin^4 x \cos^5 x \ dx$

   Solution:

$$\int \sin^4 x \cos^5 \ dx = \int u^4(1-u^2)^2 \ du = \frac{1}{9}u^9 - \frac{2}{7}u^7 + \frac{1}{5}u^5 + C$$
$$= \frac{1}{9}\sin^9 x - \frac{2}{7}\sin^7 x + \frac{1}{5}\sin^5 x + C$$

Lecture Notes: The key point here is that this works because the power of $\cos x$ is odd and we can break a $\cos x$ off to use for du.

2. Integrate with $u = \cos x$: $\int \sin^3 x \cos^5 x \ dx$

   Solution:

$$\int \sin^3 x \cos^5 x \ dx = \int (1-u^2)u^5 \ du = \frac{1}{6}u^6 - \frac{1}{8}u^8 + C$$
$$= \frac{1}{6}\cos^6 x - \frac{1}{8}\cos^8 x + C$$

Lecture Notes: The key point here is that this works because the power of $\sin x$ is odd and we can break a $\sin x$ off to use for du and the make everything else in terms of $\cos x$.

Because the power of $\cos x$ is also odd, we could also use $u = \sin x$ as in the previous problem.

3. Integrate with $u = \tan x$: $\int \tan x \ \sec^6 x \ dx$

   Solution:

$$\int \tan x \ \sec^6 x \ dx = \int \tan x \ \sec^4 x \ \sec^2 x \ dx = \int \tan x \ (1+\tan^2 x)^2 \ \sec^2 x \ dx = \int u(1+u^2)^2 \ du$$
$$= \frac{1}{6}(1+u^2)^3 + C = \frac{1}{6}(1+\tan^2 x)^3 + C$$

Lecture Notes: The substitution $u = \tan x$ works because there is an even power of $\sec x$ to peel off and use for du. This will work whenever the power of $\sec x$ is even.

4. Integrate with $u = \sec x$: $\int \tan x \ \sec^6 x \ dx$

   Solution:

$$\begin{aligned}\int \tan x \ \sec^6 x \ dx &= \int \sec^5 x(\sec x \ \tan x) \ dx = \int u^5 \ du \\ &= \frac{1}{6}u^6 + C = \frac{1}{6}\sec^6 x + C\end{aligned}$$

Lecture Notes: The substitution $u = \sec x$ works if we can peel off a $\sec x \ \tan x$ to use for du. We can do this whenever there is a spare $\tan x$, in other words when the power of $\tan x$ is odd.

5. Integrate using the product to sum formulas above: $\int \sin(2x)\sin(17x) \ dx$

   Solution:

$$\begin{aligned}\int \sin(2x)\sin(17x) \ dx &= \int \frac{1}{2}\left[\cos((2-17)x) - \cos((2+17)x)\right] \ dx \\ &= \int \frac{1}{2}\left[\cos(15x) - \cos(19x)\right] \ dx \quad \text{(Why } \cos(15x) \text{ and not } \cos(-15x)\text{?)} \\ &= \frac{1}{30}\sin(15x) - \frac{1}{38}\sin(19x) + C\end{aligned}$$

Class Problems

Lecture Notes: **Trig integrals of the form** $\int \sin^m x \ \cos^n x \ dx$

If n is odd: Break off one of the powers of $\cos x$, leaving an even power of $\cos x$. Transform these into $\sin x$ using $\cos^2 x = 1 - \sin^2 x$. Then use $u = \sin x$, $du = \cos x \ dx$ and it all works easily.
If m is odd: As above but break off one of the powers of $\sin x$. Use $u = \cos x$.
If m and n are both even: In this case use double angle formulas to reduce the powers:
$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
Examples:

$$\int \sin^2 x \cos^3 x \ dx = -\frac{3}{15}\sin^5 x + \frac{1}{3}\sin^3 x + C$$

$$\int \sin^3 x \cos^2 x \ dx = \frac{3}{15}\cos^5 x - \frac{1}{3}\cos^3 x + C$$

$$\begin{aligned}\int \sin^2 x \cos^2 x \ dx &= \int \frac{1}{2}(1 - \cos 2x) \cdot \frac{1}{2}(1 + \cos 2x) \ dx \\ &= \int \frac{1}{4}(1 - \cos^2 2x) \ dx \\ &= \int \left(\frac{1}{4} - \frac{1}{4} \cdot \frac{1}{2}(1 + \cos 4x)\right) \ dx \\ &= \int \left(\frac{1}{4} - \frac{1}{8}(1 + \cos 4x)\right) \ dx \\ &= \int \left(\frac{1}{8} - \frac{1}{8}\cos 4x\right) \ dx = \frac{1}{8}x - \frac{1}{32}\sin 4x + C\end{aligned}$$

More examples:

$$\int \sin^4 x \, \cos^5 x \, dx = \frac{35\sin^9 x - 90\sin^7 x + 63\sin^5 x}{315} + C$$
$$\int \sin^5 x \, \cos^4 x \, dx = \frac{-35\cos^9 x + 90\cos^7 x - 63\cos^5 x}{315} + C$$
$$\int \sin^4 x \, \cos^6 x \, dx = \frac{\sin(8x)}{2048} - \frac{\sin(4x)}{256} + \frac{\sin^5(2x)}{320} + \frac{3x}{256} + C$$

Lecture Notes: Trig integrals of the form $\int \tan^m x \, \sec^n x \, dx$

If n is even: Save a factor of $\sec^2 x$ and convert all other $\sec x$ to $\tan x$ using $\sec^2 x = 1 + \tan^2 x$. The use $u = \tan x$.
If m odd: Save a factor of $\sec x \tan x$, convert all other $\tan x$ into $\sec x$ (using identity) and then select $u = \sec x$.
Others: Do something ad-hoc.
Examples

$$\begin{aligned}\int \tan^3 x \sec^4 x \, dx &= \int u^3(1+u^2)\, du \qquad (u = \tan x)\\ &= \frac{1}{4}u^4 + \frac{1}{6}u^6 + C\\ &= \frac{1}{4}(\tan x)^4 + \frac{1}{6}(\tan x)^6 + C\end{aligned}$$

$$\begin{aligned}\int \tan^3 x \sec^4 x \, dx &= \int (\tan^2 x \sec^3 x)(\tan x \sec x)\, dx = \int (u^2-1)u^3 \, du \qquad (u = \sec x)\\ &= \frac{1}{6}u^6 - \frac{1}{4}u^4 + C\\ &= \frac{1}{6}\sec^6 x - \frac{1}{4}\sec^4 x + C\end{aligned}$$

Lecture Notes: Trig integrals of the form $\int \sin(mx)\cos(nx)\, dx$ (and similar integrals)

For these use the trig identities

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$
$$\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$$
$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

Examples:

$$\begin{aligned}\int \sin 2x \cos 3x \, dx &= \int \frac{1}{2}\left[\sin(5x) + \sin(-x)\right]\\ &= \frac{1}{2}\int \left[\sin(5x) - \sin(x)\right]\\ &= -\frac{1}{10}\cos(5x) + \frac{1}{2}\cos x + C\end{aligned}$$

6. **Clicker** Identify the strategy to integrate the integrals below
   Answers for all parts of this question

   A. Let $u = \sin x$.

B. Let $u = \cos x$
C. Let $u = \tan x$
D. Let $u = \sec x$
E. Some other trig identity, something tricky or maybe its impossible.

(a) $\displaystyle\int \sin^4 x\ \cos^5 x\ dx = \frac{35\sin^9 x - 90\sin^7 x + 63\sin^5 x}{315} + C$

Solution: A: Let $u = \sin x$.

(b) $\displaystyle\int \cos^2 x\ dx = \frac{1}{4}\sin 2x + \frac{1}{2}x + C$

Solution: E: half angle identity

(c) $\displaystyle\int \sec^4 x\ dx = \frac{1}{3}\tan^3 x + \tan x + C$

Solution: C: Let $u = \tan x$

(d) $\displaystyle\int \tan^4 x\ dx = \int \tan^2 x(\sec^2 -1)\ dx = \int \tan^2 x \sec^2\ dx - \int \tan^2 x\ dx$

$\displaystyle= \int \tan^2 x \sec^2\ dx - \int (\sec^2 x - 1)\ dx = \frac{1}{3}\tan^3 x - \tan x + x + C$

Solution: C: Let $u = \tan x$

(e) $\displaystyle\int \sin^5 x\ \cos^4 x\ dx = \frac{-35\cos^9 x + 90\cos^7 x - 63\cos^5 x}{315}$

Solution: B: Let $u = \cos x$.

(f) $\displaystyle\int \sin^4 x\ \cos^6 x\ dx = \frac{\sin(8x)}{2048} - \frac{\sin(4x)}{256} + \frac{\sin^5(2x)}{320} + \frac{3x}{256} + C$

Solution: E: half angle identities

7. Compute the integrals in Problem 6

8. Below are a variety of trig integral problems. Some are routine, some are tedious, and some (I think) are more difficult.

(a) $\displaystyle\int \sin^3 x \cos^2 x\ dx$

Solution: Use $u = \cos x$

$\displaystyle= \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C$

(b) $\displaystyle\int \sin^2 x \cos^7 x\ dx$

Solution: Use $u = \sin x$

$\displaystyle= -\frac{1}{9}\sin^9 x + \frac{3}{7}\sin^7 x - \frac{3}{5}\sin^5 x + \frac{1}{3}\sin^3 x + C$

(c) $\displaystyle\int \sin^4 x\ dx$

Solution: Use half angle formulas.

$\displaystyle= \frac{1}{32}\sin(4x) - \frac{1}{4}\sin(2x) + \frac{3}{8}x + C$

(d) $\displaystyle\int \sin^6 x\ \cos^2 x\ dx$

Solution: Use half angle formulas.

$\displaystyle= -\frac{1}{1024}\sin(8x) - \frac{1}{128}\sin(4x) - \frac{1}{48}\sin^3(2x) + \frac{5}{128}x + C$

(e) $\displaystyle\int \sec^3 x\ dx$

Solution: Use integration by parts. Let $u = \sec x$.

$u = \sec x$, $dv = \sec^2 x\ dx$

$du = \sec x \tan x\ du$, $v = \tan x$

$$\begin{aligned}\int \sec^3 x\ dx &= \sec x \tan x - \int \sec x \tan^2 x\ dx \\ &= \sec x \tan x - \int \sec x(\sec^2 x - 1)\ dx \\ &= \sec x \tan x - \int \sec^3 x\ dx + \int \sec x\ dx\end{aligned}$$

$$\int \sec^3 x\ dx = \frac{1}{2}\left(\sec x \tan x + \int \sec x\ dx\right)$$

(f) $\int \sec^4 x \, dx$

Solution: $u = \tan x$, $= \frac{1}{3}\tan^3 x + \tan x + C$

(g) $\int \tan^2 x \, dx$

Solution: Convert to $\sec x$ and then its easy: $= \int(\sec^2 x - 1)\, dx = \tan x - x + C$

(h) $\int \tan^4 x \, dx$

Solution: Manipulate and use $u = \tan x$: $= \int \tan^2 x(\sec^2 - 1)\, dx = \int \tan^2 x \sec^2 \, dx - \int \tan^2 x \, dx$
$= \int \tan^2 x \sec^2 \, dx - \int(\sec^2 x - 1)\, dx = \frac{1}{3}\tan^3 x - \tan x + x + C$

(i) $\int \dfrac{\cos^2 x}{1 + \sin x}\, dx$

Solution: This is really just an easy trick

$$\int \frac{\cos^2 x}{1+\sin x}\, dx = \int \frac{1-\sin^2 x}{1+\sin x}\, dx = \int \frac{(1-\sin x)(1+\sin x)}{1+\sin x}\, dx = \int 1 - \sin x \, dx = x + \cos x + C$$

(j) $\int (\sec^2 x)\sqrt{5 + \tan x}\, dx$

Solution: Just use $u = 5 + \tan x$.

$$\int (\sec^2 x)\sqrt{5+\tan x}\, dx = \int \sqrt{u}\, du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(\tan x)^{3/2} + C$$

(k) $\int \dfrac{\cos^5 x \, \sin x}{1 - \sin^2 x}\, dx$

Solution: Just convert the denominator to $\cos x$ and cancel

$$\int \frac{\cos^5 x \, \sin x}{1-\sin^2 x}\, dx = \int \frac{\cos^5 x \, \sin x}{\cos^2 x}\, dx = \int \cos^3 x \, \sin x \, dx = -\frac{1}{6}\cos^6 x + C$$

(l) $\int \dfrac{\sin 2x}{\cos^4 x + \sin^4 x}\, dx$

Solution: First step is to make the argument of sin and cos all the same. You can either make them all x's (probably will work) or all $2x$'s (what we'll do because it gives us the added benefit of reducing the power on the trig functions in denominator):

$$\begin{aligned}
\int \frac{\sin 2x}{\cos^4 x + \sin^4 x}\, dx &= \int \frac{\sin 2x}{[\cos^2 x]^2 + [\sin^2 x]^2}\, dx = \int \frac{\sin 2x}{[(1/2)(1+\cos(2x)]^2 + [(1/2)(1-\cos(2x)]^2}\, dx \\
&= \int \frac{\sin 2x}{(1/4)[1 + 2\cos(2x) + \cos^2(2x)] + (1/4)[1 - 2\cos(2x) + \cos^2(2x)]}\, dx \\
&= \int \frac{4\sin 2x}{2 + 2\cos^2(2x)}\, dx = \int \frac{2\sin 2x}{1 + \cos^2(2x)}\, dx \\
&= \int -\frac{1}{1+u^2}\, du \qquad (u = \cos(2x)) \\
&= -\arctan(u) + C = -\arctan(\cos(2x)) + C
\end{aligned}$$