Math 132 - October 4, 2017

Solutions

Sections Covered: 7.3

Trig Idenities:

- $\sin^2\theta + \cos^2\theta = 1$
- $1 + \tan^2\theta = \sec^2\theta$
- $1 + \cot^2\theta = \csc^2\theta$
- $\sin(A \pm B) = \sin A\cos B \pm \sin B\cos A$
- $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$
- $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$
- $\sin A\sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
- $\cos A\cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$
- $\sin A\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$
- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
- $\sin(2\theta) = 2\sin\theta\cos\theta$
- $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

Trig Integral Stategies

- $\int \sin^m x\ \cos^n x\ dx$
- $\int \tan^m x\ \sec^n x\ dx$
- $\int \tan x\ dx$
- $\int \sin(nx)\ \cos(mx)\ dx$
- $\int \sec x\ dx$
- $\int \sec^3 x\ dx$
- Others

Warm-up Problems

1. **Clicker** $\int \sin x\cos x\ dx$ =??

   (a) $\frac{1}{2}\sin^2 x + C$

   (b) $-\frac{1}{2}\cos^2 x + C$

   (c) Both A and B

   (d) Neither A nor B

   (e) Anything you want it to be

   Solution: This is a simple substitution. You can use either $u = \sin x$ or $u = \cos x$. In either case you get

   $$\begin{aligned}\int \sin x\cos x\ dx &= \int u\ du \qquad (u = \sin x)\\ &= \frac{u^2}{2} + C = \frac{1}{2}\sin^2 x + C\\ \int \sin x\cos x\ dx &= -\int u\ du \qquad (u = \cos x)\\ &= -\frac{u^2}{2} + C = -\frac{1}{2}\cos^2 x + C\end{aligned}$$

   Note that since $\sin^2 x + \cos^2 x = 1$ we have

   $$\begin{aligned}\int \sin x\cos x\ dx &= \int u\ du \qquad (u = \sin x)\\ &= \frac{u^2}{2} + C = \frac{1}{2}\sin^2 x + C\\ &= \frac{1}{2}(1 - \cos^2 x) + C = -\frac{1}{2}\cos^2 x + \frac{1}{2} + C\end{aligned}$$

   So the answers really just differ by a constant.

2. Find the derivatives:

   (a) $\frac{d}{dx}(\sec x + \tan x)$

   Solution: $\sec x\tan x + \sec^2 x = \sec x(\tan x + \sec x)$

   (b) $\frac{d}{dx}(\csc x + \cot x)$

   Solution: $-\csc x\cot x - \csc^2 x = \csc x(-\cot x + \csc x)$

Lecture Notes: A note on integration by parts, the "normal" types you will see and you have to have "down cold":

- $\int (\text{Polynomial}(x))e^x \, dx$
- $\int (\text{Polynomial}(x)) \sin x \, dx$
- $\int (\text{Polynomial}(x)) \cos x \, dx$
- $\int e^x \sin x \, dx$
- $\int e^x \cos x \, dx$
- $\int \ln x \, dx$

3. Use integration by parts.
$\int \sec^3 x \, dx$

Solution:

$u = \sec x$, $dv = \sec^2 x \, dx$
$du = \sec x \tan x \, du$, $v = \tan x$

$$\begin{aligned}\int \sec^3 x \, dx &= \sec x \tan x - \int \sec x \tan^2 x \, dx \\ &= \sec x \tan x - \int \sec x(\sec^2 x - 1) \, dx \\ &= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx \\ \int \sec^3 x \, dx &= \frac{1}{2}\left(\sec x \tan x + \int \sec x \, dx\right)\end{aligned}$$

Class Problems

4. Compute $\int \sec x \, dx$

Solution: Use $u = \tan x + \sec x$:
$= \int \sec x \left(\frac{\tan x + \sec x}{\tan x + \sec x}\right) \, dx = \ln|\tan x + \sec x| + C$

5. Compute $\int \csc x \, dx$

Solution: Use $u = \cot x + \csc x$:
$= \int \csc x \left(\frac{\cot x + \csc x}{\cot x + \csc x}\right) \, dx = -\ln|\cot x + \csc x| + C$

Lecture Notes: Now that we have this, we can complete the integral of $\sec^3 x$:

$$\begin{aligned}\int \sec^3 x \, dx &= \frac{1}{2}\left(\sec x \tan x + \int \sec x \, dx\right) \\ &= \frac{1}{2}\sec x \tan x + \frac{1}{2}\int \sec x \, dx \\ &= \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\tan x + \sec x| + C\end{aligned}$$

Lecture Notes: Trig integrals of the form $\int \sin^m x \, \cos^n x \, dx$

If n is odd: Break off one of the powers of $\cos x$, leaving an even power of $\cos x$. Transform these into $\sin x$ using $\cos^2 x = 1 - \sin^2 x$. Then use $u = \sin x$, $du = \cos x \, dx$ and it all works easily.

If m is odd: As above but break off one of the powers of $\sin x$. Use $u = \cos x$.

If m and n are both even: In this case use double angle formulas to reduce the powers:
$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

Examples:

$$\int \sin^2 x \cos^3 x \, dx = -\frac{3}{15}\sin^5 x + \frac{1}{3}\sin^3 x + C$$

$$\int \sin^3 x \cos^2 x \, dx = \frac{3}{15}\cos^5 x - \frac{1}{3}\cos^3 x + C$$

$$\begin{aligned}\int \sin^2 x \cos^2 x \, dx &= \int \frac{1}{2}(1 - \cos 2x) \cdot \frac{1}{2}(1 + \cos 2x) \, dx \\ &= \int \frac{1}{4}(1 - \cos^2 2x) \, dx \\ &= \int \left(\frac{1}{4} - \frac{1}{4} \cdot \frac{1}{2}(1 + \cos 4x)\right) \, dx \\ &= \int \left(\frac{1}{4} - \frac{1}{8}(1 + \cos 4x)\right) \, dx \\ &= \int \left(\frac{1}{8} - \frac{1}{8}\cos 4x\right) \, dx = \frac{1}{8}x - \frac{1}{32}\sin 4x + C\end{aligned}$$

6. **Clicker** Identify the strategy to integrate
$\int \sin^4 x \, \cos^5 x \, dx$

   (a) Use $\cos^2 x = 1 - \sin^2 x$ and then let $u = \sin x$. **Correct**
   (b) Use $\sin^2 x = 1 - \cos^2 x$ and then let $u = \cos x$.
   (c) Use half angle identities to reduce the even powers of $\sin x$ and $\cos x$.
   (d) Give up, nothing will work

   Solution: $\int \sin^4 x \, \cos^5 x \, dx = \dfrac{35\sin^9 x - 90\sin^7 x + 63\sin^5 x}{315} + C$

7. **Clicker** Identify the strategy to integrate
$\int \sin^5 x \, \cos^4 x \, dx$

   (a) Use $\cos^2 x = 1 - \sin^2 x$ and then let $u = \sin x$.
   (b) Use $\sin^2 x = 1 - \cos^2 x$ and then let $u = \cos x$. **Correct**
   (c) Use half angle identities to reduce the even powers of $\sin x$ and $\cos x$.
   (d) Give up, nothing will work

   Solution: $\int \sin^5 x \, \cos^4 x \, dx = \dfrac{-35\cos^9 x + 90\cos^7 x - 63\cos^5 x}{315}$

8. **Clicker** Identify the strategy to integrate
$\int \sin^4 x \, \cos^6 x \, dx$

   (a) Use $\cos^2 x = 1 - \sin^2 x$ and then let $u = \sin x$.

(b) Use $\sin^2 x = 1 - \cos^2 x$ and then let $u = \cos x$.

(c) Use half angle identities to reduce the even powers of $\sin x$ and $\cos x$. **Correct**

(d) Give up, nothing will work

Solution: $\displaystyle\int \sin^4 x \, \cos^6 x \, dx = \frac{\sin(8x)}{2048} - \frac{\sin(4x)}{256} + \frac{\sin^5(2x)}{320} + \frac{3x}{256} + C$

9. Actually compute the integrals above.

Lecture Notes: Trig integrals of the form $\displaystyle\int \tan^m x \, \sec^n x \, dx$

If n is even: Save a factor of $\sec^2 x$ and convert all other $\sec x$ to $\tan x$ using $\sec^2 x = 1 + \tan^2 x$. The use $u = \tan x$.

If m odd: Save a factor of $\sec x \tan x$, convert all other $\tan x$ into $\sec x$ (using identity) and then select $u = \sec x$.

Others: Do something ad-hoc.

Examples

$$\begin{aligned}
\int \tan^3 x \sec^4 x \, dx &= \int u^3(1+u^2)\, du \qquad (u = \tan x) \\
&= \frac{1}{4}u^4 + \frac{1}{6}u^6 + C \\
&= \frac{1}{4}(\tan x)^4 + \frac{1}{6}(\tan x)^6 + C \\
\int \tan^3 x \sec^4 x \, dx &= \int (\tan^2 x \sec^3 x)(\tan x \sec x)\, dx = \int (u^2-1)u^3 \, du \qquad (u = \sec x) \\
&= \frac{1}{6}u^6 - \frac{1}{4}u^4 + C \\
&= \frac{1}{6}\sec^6 x - \frac{1}{4}\sec^4 x + C
\end{aligned}$$

Lecture Notes: Trig integrals of the form $\displaystyle\int \sin(mx)\cos(nx)\, dx$ (and similar integrals)

For these use the trig identities

$$\begin{aligned}
\sin A \sin B &= \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right] \\
\cos A \cos B &= \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right] \\
\sin A \cos B &= \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]
\end{aligned}$$

Examples:

$$\begin{aligned}
\int \sin 2x \cos 3x \, dx &= \int \frac{1}{2}\left[\sin(5x) + \sin(-x)\right] \\
&= \frac{1}{2}\int \left[\sin(5x) - \sin(x)\right] \\
&= -\frac{1}{10}\cos(5x) + \frac{1}{2}\cos x + C
\end{aligned}$$