Math 132: Discussion Session: Week 6

Directions: In groups of 3-4 students, work the problems on the following page. Below, list the members of your group and your answers to the specified questions. Turn **this paper** in at the end of class. You do not need to turn in the question page or your work.

Additional Instructions: It is okay if you do not completely finish all of the problems. Also, each group member should work through each problem, as similar problems may appear on the exam.

Scoring:

Correct answers	Grade
0–3	0%
4–6	80%
7–11	100%

Group Members:

6.3: The cylindrical shell method.

(1) The volume is:

(2) The volume is:

6.5: Average Value.

(1) The average value is:

(2) The average value is:

7.1: Integration by Parts.

(1) $\displaystyle\int_0^1 xe^x\,dx =$

(2) $\displaystyle\int x^3 \ln x\,dx =$

(3) $\displaystyle\int \ln\sqrt{x}\,dx =$

(4) $\displaystyle\int (\ln x)^2\,dx =$

(5) $\displaystyle\int_{-\pi/3}^{\pi/3} x^4 \sin x\,dx =$

(6) $\displaystyle\int_0^{\pi} e^x \sin x\,dx =$

(7) $\displaystyle\int e^{\sqrt[3]{x}}\,dx =$

Math 132 Discussion Session: Week 6

6.3: The cylindrical shell method. Using the cylindrical shell method, find the volume of the solid obtained by rotating the region enclosed by the given curves about the given axis.

(1) $y = a - x$ with $a > 0$, $x = 0$, $y = 0$, about $x = -1$. Your answer should be in terms of a.

Solution: We use vertical cyclinders centered around the line $x = -1$. The line $y = a - x$ intersects $y = 0$ at $x = a$, so the cylinders range from $x = 0$ to $x = a$. The radius of the cylindrical shell at x is $x + 1$. Its height is $a - x$. Thus, the area of the cylindrical shell is

$$A(x) = 2\pi(x + 1)(a - x).$$

The volume of the solid of revolution is

$$\begin{aligned}
\int_0^a A(x)\,dx &= 2\pi \int_0^a (x+1)(a-x)\,dx \\
&= 2\pi \int_0^a \left(a + (a-1)x - x^2\right)\,dx \\
&= 2\pi \left(ax + \tfrac{1}{2}(a-1)x^2 - \tfrac{1}{3}x^3\right)\Big|_0^a \\
&= 2\pi(a^2 + \tfrac{1}{2}(a-1)a^2 - \tfrac{1}{3}a^3) \\
&= 2\pi(a^2 + \tfrac{1}{2}a^3 - \tfrac{1}{2}a^2 - \tfrac{1}{3}a^3) \\
&= \boxed{\pi a^2 + \tfrac{\pi}{3}a^3.}
\end{aligned}$$

(2) $x = y(4 - y)$, $x = 0$, about the x-axis

Solution: We use horizontal cylinders centered around the x-axis. The radius of the cylinder at y is just y, and its height is $y(4 - y)$. Thus, the area of each cylinder is

$$A(y) = 2\pi y(y(4 - y)) = 8\pi y^2 - 2\pi y^3.$$

Since $x = y(4 - y)$ intersects $x = 0$ when $y = 0$ or $y = 4$, our cylindrical shells are at y-values in the range $0 \le y \le 4$. Thus, the volume of our solid is

$$\begin{aligned}
\int_0^4 A(y)\,dy &= \int_0^4 \left(8\pi y^2 - 2\pi y^3\right)\,dy \\
&= \left(\tfrac{8\pi}{3}y^3 - \tfrac{\pi}{2}y^4\right)\Big|_0^4 \\
&= \tfrac{8\pi}{3} \cdot 64 - \tfrac{\pi}{2} \cdot 256 = \tfrac{512\pi}{3} - 128\pi = \boxed{\frac{128\pi}{3}}.
\end{aligned}$$

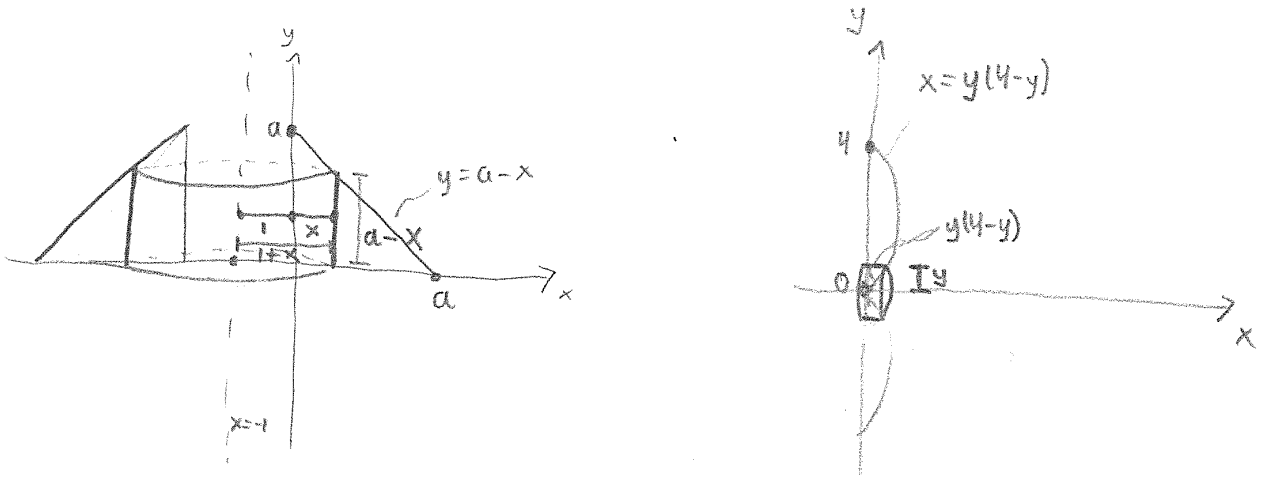

FIGURE 1. Volumes of solids of revolution using the cylindrical shell method.

6.5: Average Value.

(1) Compute the average value of $f(x) = 2x^3 - 9x^2$ on $[-1, 3]$.

Solution: The length of the interval is $3 - (-1) = 4$, so the average value is

$$\tfrac{1}{4}\int_{-1}^{3} \left(2x^3 - 9x^2\right)\, dx = \tfrac{1}{4}\left(\tfrac{1}{2}x^4 - 3x^3\right)\Big|_{-1}^{3} = \tfrac{1}{8}(81-1) - \tfrac{3}{4}(27-(-1)) = 10 - 21 = \boxed{-11}.$$

(2) Compute the average value of $g(x) = e^{nx}$ on $[-1, 1]$. (Your answer will be in terms of n.)

Solution: The length of the interval is $1 - (-1) = 2$, so we'd like to compute

$$\tfrac{1}{2}\int_{-1}^{1} e^{nx}\, dx.$$

Don't let the n frighten you. If you need to acquire bravery, you might first pick a specific value for n such as 5 or 117 and compute that $\int e^{117x} = \frac{1}{117}e^{117x} + C$. Once we are brave enough, we return to the original problem, and see that $\int e^{nx}\, dx = \frac{1}{n}e^{nx} + C$. Thus,

$$\tfrac{1}{2}\int_{-1}^{1} e^{nx}\, dx = \tfrac{1}{2}\left(\tfrac{1}{n}e^{nx}\right)\Big|_{x=-1}^{1} = \boxed{\tfrac{1}{2n}\left(e^n - e^{-n}\right)} = \boxed{\frac{\sinh n}{n}}.$$

7.1: Integration by Parts. Compute the following integrals using integration by parts:

(1) $\displaystyle\int_0^1 xe^x\, dx$

Solution: We'd rather differentiate x than integrate it, so we set

$$\begin{aligned} f(x) &= x, & g'(x) &= e^x, \\ f'(x) &= 1, & g(x) &= e^x. \end{aligned}$$

Then

$$\begin{aligned} \int_0^1 xe^x\, dx &= \int_0^1 fg'\, dx \\ &= fg|_0^1 - \int_0^1 f'g\, dx \\ &= xe^x|_0^1 - \int_0^1 e^x, dx \\ &= (e-0) - e^x|_0^1 \\ &= e - (e-1) = \boxed{1}. \end{aligned}$$

(2) $\displaystyle\int x^3 \ln x\, dx$

Solution: We'd rather differentiate x^3 than integrate x^3, but we *definitely* would rather differentiate $\ln x$ than integrate $\ln x$. So, we set

$$\begin{aligned} f'(x) &= x^3, & g(x) &= \ln x, \\ f(x) &= \tfrac{1}{4}x^4, & g'(x) &= \tfrac{1}{x}. \end{aligned}$$

Then

$$\begin{aligned}\int x^3 \ln x \, dx &= \int f'g \, dx \\ &= fg - \int fg' \, dx \\ &= \tfrac{1}{4}x^4 \ln x - \int \tfrac{1}{4}x^3 \, dx \\ &= \boxed{\tfrac{1}{4}x^4 \ln x - \tfrac{1}{16}x^4 + C}.\end{aligned}$$

(3) $\int \ln \sqrt{x} \, dx$

Solution: There are several ways to approach this problem. The quickest is to use properties of the logarithm to notice that $\ln \sqrt{x} = \frac{1}{2} \ln x$. We can compute $\int \ln x \, dx$ using integration by parts as done in class or in Example 2 in the book. Thus,

$$\int \ln \sqrt{x} \, dx = \tfrac{1}{2} \int \ln x = \boxed{\tfrac{1}{2}(x \ln x - x) + C}.$$

Without this method, we can still do a standard integration by parts with

$$\begin{aligned} f(x) &= \ln \sqrt{x}, & g'(x) &= 1 \\ f'(x) &= \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2x}, & g(x) &= x. \end{aligned}$$

Then,

$$\int \ln \sqrt{x} = x \ln \sqrt{x} - \int \frac{x}{2x} \, dx = \boxed{x \ln \sqrt{x} - \tfrac{x}{2} + C}.$$

Our two answers look different, but we can recognize them as the same using the identity $\ln \sqrt{x} = \frac{1}{2} \ln x$.

(4) $\int (\ln x)^2 \, dx.$

Solution: There are a couple integration by parts strategies here. One of them is

$$\begin{aligned} f(x) &= (\ln x)^2, & g'(x) &= 1, \\ f'(x) &= 2 \ln x \cdot \tfrac{1}{x}, & g(x) &= x. \end{aligned}$$

Then,

$$\begin{aligned}\int (\ln x)^2 \, dx &= \int fg' \, dx \\ &= fg - \int f'g \, dx \\ &= x(\ln x)^2 - \int 2 \ln x \\ &= x(\ln x)^2 - 2(x \ln x - x) = \boxed{x(\ln x)^2 - 2x \ln x + 2x + C}.\end{aligned}$$

Again, we needed to use that $\int x \ln x \, dx = x \ln x - x$, which we either know from class or from the book, or could figure out using a second integration by parts step.

Alternatively, using $\int x \ln x \, dx = x \ln x - x$, we could set up the integration by parts a different way, via

$$\begin{aligned} f(x) &= \ln x, & g'(x) &= \ln(x) \\ f'(x) &= \tfrac{1}{x}, & g(x) &= x \ln x - x. \end{aligned}$$

Then,

$$\begin{aligned}\int (\ln x)^2\,dx &= \ln x(x\ln x - x) - \int \tfrac{1}{x}(x\ln x - x)\,dx \\ &= x(\ln x)^2 - x\ln x - \int (\ln x - 1)\,dx \\ &= x(\ln x)^2 - x\ln x - ((x\ln x - x) - x) \\ &= \boxed{x(\ln x)^2 - 2x\ln x + 2x + C}.\end{aligned}$$

(5) $\displaystyle\int_{-\pi/3}^{\pi/3} x^4 \sin x\,dx$. Hint: There's a much faster way to do this problem than integration by parts.
Solution: We'd set up an integration by parts with

$$\begin{aligned} f(x) &= x^4, & g'(x) &= \sin x, \\ f'(x) &= 4x^3, & g(x) &= -\cos x. \end{aligned}$$

As a result, we'd need to compute $\int f'g\,dx = -4\int x^3\cos x\,dx$. That's better than what we started with, since x^3 is simpler than x^4, but we'd need to integrate by parts three more times to solve the problem.

Instead, following the hint, we look for something else. We note that the bounds are symmetric, and we see that $x^4\sin x$ is an odd function because $(-x)^4\sin(-x) = x^4(-\sin x) = -x^4\sin x$. Thus,

$$\int_{-\pi/3}^{\pi/3} x^4\sin x\,dx = \boxed{0}.$$

(6) $\displaystyle\int_0^{\pi} e^x\sin x\,dx$. Hint: Boldly integrate by parts twice. Then, solve an equation for $\int e^x\sin x\,dx$.
Solution: Following the hint, we integrate by parts twice. Which term should we differentiate, and which term should we integrate? In this case, either way we get something that's neither more complicated nor simpler, so we can make either choice. Let's go with

$$\begin{aligned} f'(x) &= e^x, & g(x) &= \sin x \\ f(x) &= e^x, & g'(x) &= \cos x. \end{aligned}$$

Then,

$$\begin{aligned}\int e^x\sin x &= \int f'g\,dx \\ &= fg - \int fg'\,dx \\ &= e^x\sin x - \int e^x\cos x\,dx.\end{aligned}$$

The hint tells us to integrate by parts twice. Which term should we integrate and which term should we differentiate this time? This time, one of the choices is good and one of the choices is bad. If we make the wrong choice, we'll just undo our work and get back to where we started. We need to differentiate the same term we differentiated before, and integrate the same term we integrated before.

$$\begin{aligned} f'(x) &= e^x, & g(x) &= \cos x \\ f(x) &= e^x, & g(x) &= -\sin x. \end{aligned}$$

Thus,

$$\begin{aligned}\int e^x \cos x &= \int f'g\,dx \\ &= fg - \int fg'\,dx \\ &= e^x \cos x + \int e^x \sin x\,dx.\end{aligned}$$

Putting everything together, we see that

$$\begin{aligned}\int e^x \sin x\,dx &= e^x \sin x - \left(e^x \cos x + \int e^x \sin x\,dx\right) \\ &= e^x \sin x - e^x \cos x - \int e^x \sin x\,dx.\end{aligned}$$

At this point, the hint tells us to solve an equation for $\int e^x \sin x\,dx$. Indeed, we see that we have $\int e^x \sin x\,dx$ on both sides of the equation. Isolating the variable, we see that

$$\begin{aligned}2\int e^x \sin x\,dx &= e^x \sin x - e^x \cos x, \\ \int e^x \sin x\,dx &= \tfrac{1}{2}e^x(\sin x - \cos x).\end{aligned}$$

The problem asks for a definite integral, so we finish by plugging in the bounds

$$\int_0^\pi e^x \sin x\,dx = \tfrac{1}{2}\left(e^\pi(0-(-1)) - e^0(0-1)\right) = \boxed{\tfrac{1}{2}(e^\pi + 1)}.$$

(7) $\int e^{\sqrt[3]{x}}\,dx$. Hint: First do the substitution $u = \sqrt[3]{x}$.

Solution: As the hint suggests, we should use the substitution $u = x^{1/3}$, so $du = \frac{1}{3}x^{-2/3}\,dx$. Thus, we compute

$$\begin{aligned}\int e^{\sqrt[3]{x}}\,dx &= \int e^{x^{1/3}} \cdot 3x^{2/3} \cdot \tfrac{1}{3}x^{-2/3}\,dx \\ &= \int e^u \cdot 3u^2\,du \\ &= 3\int u^2 e^u\,du.\end{aligned}$$

Now this integral seems more appropriate for integration by parts. It's clear that we'd rather differentiate u^2 than integrate it, so we set up our integration by parts as

$$\begin{aligned} f(u) &= u^2, & g'(u) &= e^u, \\ f'(u) &= 2u, & g(u) &= e^u.\end{aligned}$$

Thus, by integration by parts,

$$\int u^2 e^u\,du = u^2 e^u - \int 2ue^u\,du = u^2 e^u - 2\int ue^u\,du.$$

This integral can be handled with another integration by parts. Fortunately for us, we already did this integral earlier in the worksheet, so we know that $\int ue^u\,du = ue^u - e^u + C$. Putting everything together, we see that

$$\int u^2 e^u = u^2 e^u - 2ue^u + 2e^u + C.$$

Of course, we're note done, since we haven't answered the question. We substitute back in to see that

$$\begin{aligned}\int e^{\sqrt[3]{x}} &= 3\int u^2 e^u\,du \\ &= 3e^u(u^2-2u+2)+C \\ &= \boxed{3e^{\sqrt[3]{x}}\left(x^{2/3}-2x^{1/3}+2\right)+C.}\end{aligned}$$