Math 132 - September 22, 2017
Solutions

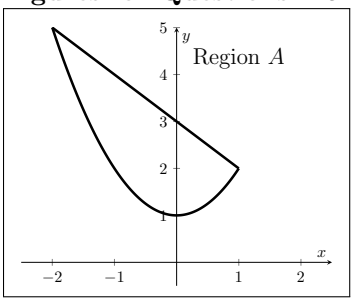
6.2-6.3: Volumes

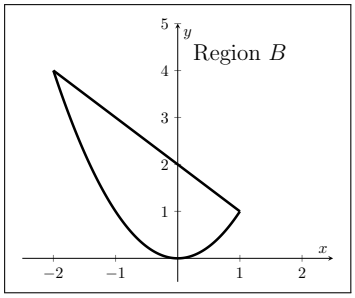
1. Is the volume a volume of rotation? If not you probably need: $V = \int_a^b A(x)\,dx$ or $V = \int_a^b A(y)\,dy$
2. Volume of revolution? Washers or shells? Where is the axis of rotation? Did you draw the solid (or the part of the solid in the xy-plane)?
3. Washers: $V = \displaystyle\int_a^b \pi r^2\,dx$ or $V = \displaystyle\int_a^b \pi r^2\,dy$
   - Identify if integral is dx or dy. (If axis is vertical, you should integrate dy. If axis is horizontal, you should integrate dx.)
   - Identify outer and inner radii in your picture (make sure you measure from the axis to the "original" region, not the reflected region)
   - Set up integral and integrate
4. Shells: $V = \int_a^b 2\pi r h\,dr$
   - Identify if the integral is dx or dy. (If axis is vertical, you should integrate dx. If axis is horizontal, you should integrate dy.)
   - Identify the radius, measured to your original region. Identify the height.
   - Set up the integral and integrate.

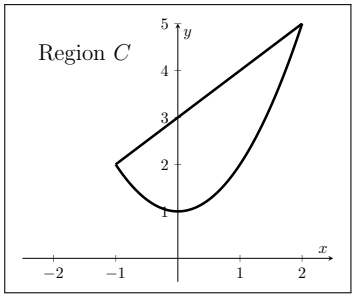
Warm-up Problems

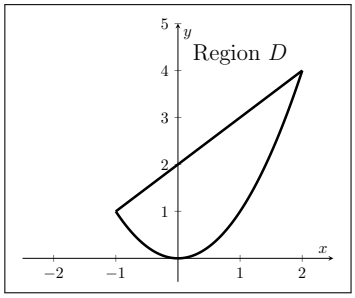
1. **Clicker** Given two y values, y_1 and y_2, how do you find the distance between these y-values?
   (a) $y_1 - y_2$ (b) $y_2 - y_1$ (c) $y_1 + y_2$ (d) $|y_1 - y_2|$ **Correct** (e) $|y_2 - y_1|$ **Correct**
   Solution: Note that answers (a) and (b) are only correct half the time.

Figures for Questions 2-5









2. Let R_1 be the region bounded by $y = x^2$ and $y = x + 2$. Find the region above that matches R_1.
   (a) A (b) B (c) C (d) D **Correct** (e) None
3. Let R_2 be the region bounded by $y = x^2$ and $y = -x + 2$. Find the region above that matches R_2.
   (a) A (b) B **Correct** (c) C (d) D (e) None
4. Let R_3 be the region bounded by $y = x^2 + 1$ and $y = x + 3$. Find the region above that matches R_3.
   (a) A (b) B (c) C **Correct** (d) D (e) None
5. Let R_4 be the region bounded by $y = x^2 + 1$ and $y = -x + 3$. Find the region above that matches R_4.
   (a) A **Correct** (b) B (c) C (d) D (e) None

Class Problems

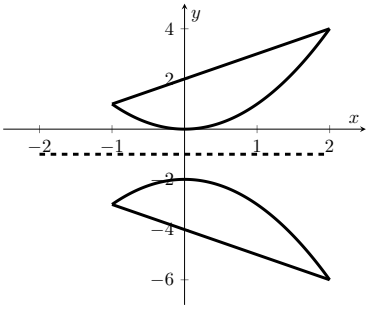
Lecture Notes: Follow the steps outlined in the summary box on top. Here are some key points to notice.

- Draw your region, draw the axis.

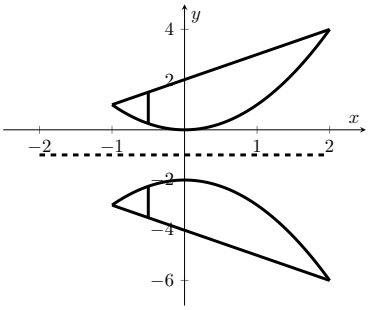
- I also like to draw the region reflected across the axis and sometimes I like to draw a few circles to show the rotation.
- Always measure from your axis to your original region. NOT to the reflected region.
- Decide if you are working with dx or dy.
- Find the inner and outer radii as expressions of x or y (depending on if you are doing dx or dy).

6. Take the region bounded by $y = x^2$ and $y = x + 2$.

   (a) Draw this region rotated about $y = -1$
   Solution:



   (b) Draw in a slice when $x = -1/2$.
   Solution:



   (c) Draw in more slices: $x = -1$, $x = 0$, $x = 1$, $x = 2$.

   (d) With your fingers on your diagram, point out the outer radius of the washer. For a given slice at x, write down a formula for the outer radius.
   Solution: $r_o = 1 + y = 1 + (x + 2)$

   (e) With your fingers on your diagram, point out the inner radius of the washer. For a given slice at x, write down a formula for the inner radius.
   Solution: $r_i = 1 + y = 1 + (x^2)$

   (f) With your fingers, point out the x values for which you have slices.
   Solution: $x = -1$ to $x = 2$

   (g) Cross sectional area is $A(x) = \pi(r_o^2 - r_i^2)$. Find an expression for $A(x)$.
   Solution: $A(x) = (x + 3)^2 - (x^2 + 1)^2$

   (h) Write down an integral representing the volume.
   Solution: $V = \pi \displaystyle\int_{-1}^{2} \left[(x + 3)^2 - (x^2 + 1)^2\right]\, dx = \frac{117}{5}\pi$

7. **Clicker** Repeat Problem 6 but this time rotate about $y = 0$ (the x-axis). In particular, click on the cross sectional area:

   (a) $A(x) = \left[(2 + x)^2 - (x^2)^2\right]$

   (b) $A(x) = \pi\left[(2 + x)^2 - (x^2)^2\right]$ **Correct**

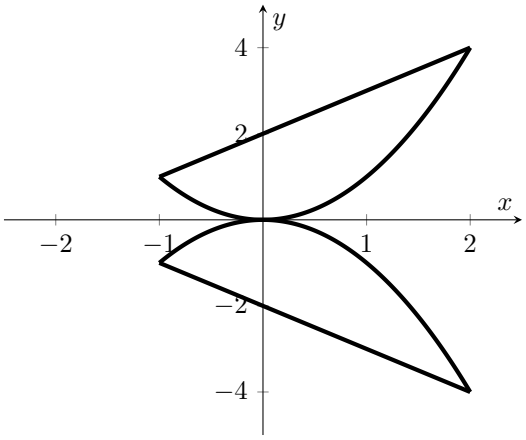
(c) $A(x) = \pi\left[(2+x) - (x^2)\right]^2$

(d) $A(x) = \pi\left[(2+x) + (x^2)\right]^2$

(e) $A(x) = \pi\left[(2+x)^2 + (x^2)^2\right]$

(f) I hate volumes!!

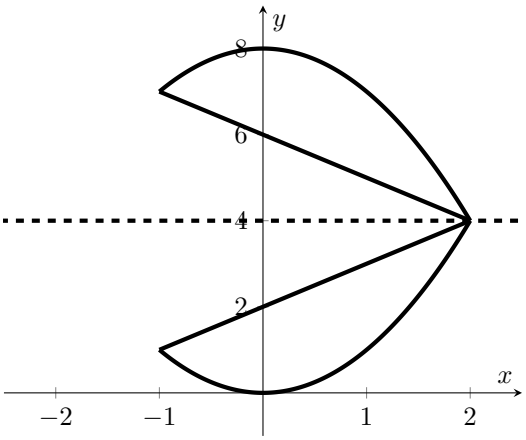
Solution:



$V = \pi \int_{-1}^{2} \left[(2-x)^2 - (x^2)^2\right]\ dx = \frac{12}{5}\pi$

8. Repeat Problem 6 but this time rotate about $y = 4$.

Solution:



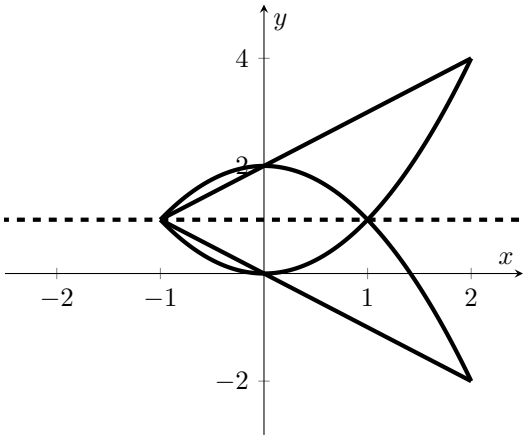
$V = \pi \int_{-1}^{2} \left[(4-x^2)^2 - (2-x)^2\right]\ dx = \frac{108}{5}\pi$

9. Repeat Problem 6 but this time rotate about $y = 100$.

Solution: $V = \pi \int_{-1}^{2} \left[(100-x^2)^2 - (100-(x+2))^2\right]\ dx$

10. Repeat Problem 6 but this time rotate about $y = 1$. (Tricky!)

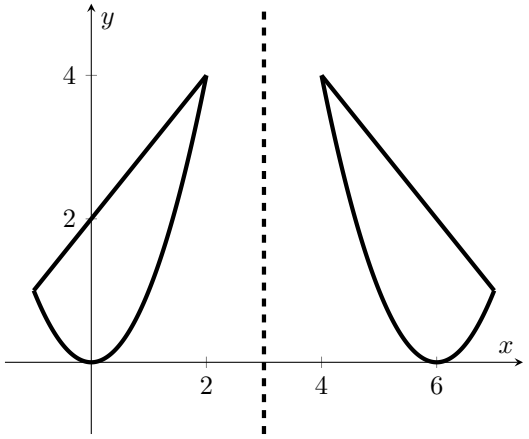
Solution:



$$\begin{aligned} V =& \pi \int_{-1}^{0} (1-x^2)^2 \, dx + \pi \int_{0}^{1} (x+2-1)^2 \, dx + \pi \int_{1}^{2} \left[(x+2-1)^2 - (x^2-1)^2 \right] \, dx \\ =& \left(\frac{8}{15} + \frac{7}{3} + \frac{19}{5} \right) \pi = \frac{20}{3} \pi \end{aligned}$$

11. Repeat Problem 6 but this time rotate about $x = 3$. (Tricky!)

Solution:



It is really important to notice the different slices for $0 \le y \le 1$ and $1 \le y \le 4$.

$$\begin{aligned} V =& \pi \int_{0}^{1} \left[(3+\sqrt{y})^2 - (3-\sqrt{y})^2 \right] \, dy + \pi \int_{1}^{4} \left[(3-(y-2))^2 - (3-\sqrt{y})^2 \right] \, dy \\ =& 8\pi + \frac{29}{2} \pi = \frac{45}{2} \pi \end{aligned}$$

There is a better choice for these volumes: shells. We will explore these next time.

12. Do all these problems, but with the other regions in Problems 2-5.