Math 132 - September 20, 2017
Solutions

6.2-6.3: Volumes

- Volumes by slicing: $V = \int_a^b A(x)\,dx$ (or $A(y)\,dy$)
- Volumes of revolution (shells): $V = \int_a^b 2\pi rh\,dr$
- Volumes of revolution (washers): $V = \int_a^b \pi r^2\,dx$ or $V = \int_a^b \pi r^2\,dy$

Warm-up Problems

1. **Clicker** I have done the following:

   I. Looked at my exam and determined where I made mistakes.

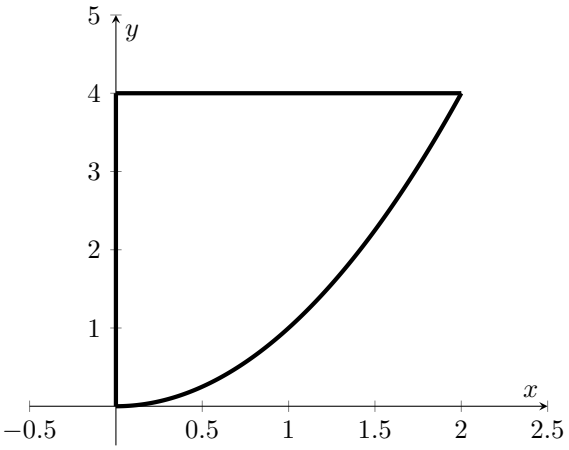
   II. Looked at the histogram of exam scores on line to see where I stand compared to the rest of the class.

   III. If I did poor on the exam, I have developed a plan to do better on the next test.
   (Hint: if you didn't do so well on the exam "work harder" is usually not a plan that works. Most students need a better plan than this.)

   (a) Yes (b) No (c) What???

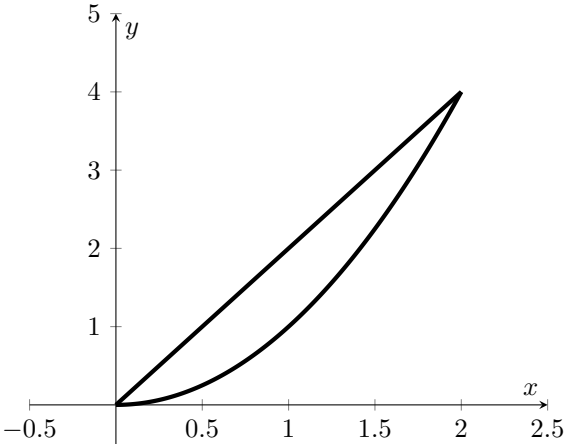
2. Let R_1 be the region in quadrant I, between $y = x^2$, $x = 0$ and $y = 4$. Graph this region.

   Solution:



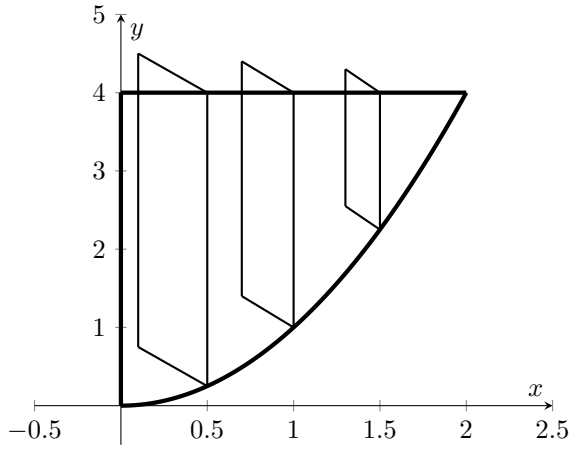
3. Let R_2 be the region between $y = x^2$ and $y = 2x$. Graph this region.

   Solution:



4. Imagine a mystery solid with a base R_1 from Problem 2 sitting on a cutting board. You take out your large knife/sword and make cuts perpendicular to the x-axis. Each of these cuts results in a square. Draw the solid.

   Solution:



Lecture Notes: To find the volume of the solid, we just consider chopping up the solid as described. It is not completely necessary to be able to draw the solid, but it is usually necessary to draw the base (and anything in 2-space is usually pretty easy to draw).

A small chunk of volume is given by $A(x)\,\Delta x$ (draw it to understand why!).

Volume Rule Let S be a solid lying between $x = a$ and $x = b$ with the area of cross sections perpendicular to the x-axis given by $A(x)$.

$$\text{Volume } S = \int_a^b A(x)\,dx$$

For any given x, we need a formula for $A(x)$. In this case:

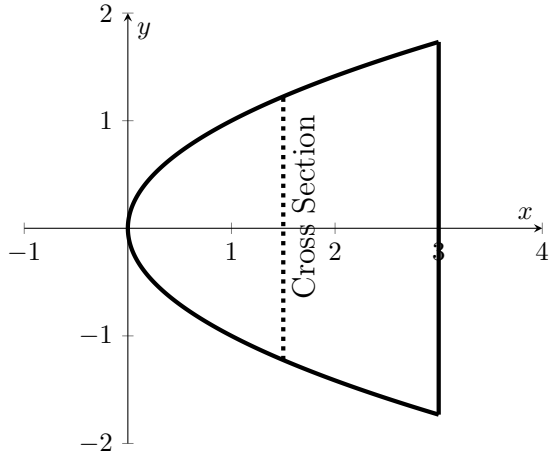
$$\begin{aligned} A(x) =& (\text{height in } xy \text{ plane}) \cdot (\text{height coming out of } xy \text{ plane}) \\ =& (4-x^2)(4-x^2) = (4-x^2)^2 \end{aligned}$$

And the volume is

$$V = \int_0^2 A(x)\,dx = \int_0^2 (4-x^2)^2\,dx = \frac{256}{15}$$

Class Problems

5. **Clicker** Let S be a solid with base enclosed by $x = y^2$ and $x = 3$.
Vertical cross sections of S are squares perpendicular to the x-axis.



Find a formula for $A(x)$, the cross sectional area at x.

(a) 4

(b) x

(c) $4x$ **Correct**

(d) x^2

(e) $4y^2$

Solution: $A = (2y)^2 = (2\sqrt{x})^2 = 4x$

6. Same as in Problem 5 except cross sections are semi-circles. Find $A(x)\pi y^2 = \pi x$.

7. Same as in Problem 5 except cross sections are equilateral triangles. Find $A(x)\frac{\sqrt{3}}{4}(2y^2) = \frac{\sqrt{3}}{2}x$.

8. Find the volumes in the previous problems.

   Solution: Problem 5: $V = \int_0^3 A(x)\,dx = \int_0^3 4x\,dx = 18$

   Problem 6: $V = \int_0^3 A(x)\,dx = \int_0^3 \pi x\,dx = \frac{9\pi}{2}$

   Problem 5: $V = \int_0^3 A(x)\,dx = \int_0^3 \frac{\sqrt{3}}{2}x\,dx = \frac{9\sqrt{3}}{4}$

9. What do the horizontal cross sections look like in the Problems 5, 6 and 7?

   Solution: This is a very difficult question. Visualizing these solids is difficult, but it is still helpful to try and visualize. And, it is a nice challenge to think about. For now, we're going to leave this at "I don't know.". The key is that the math can be done while visualizing only in the plane and drawing the easy cross sections. If the cross sections aren't easy, then try slicing in a different direction.

Lecture Notes: The other type of volume problems we are going to see are volumes of revolution. Basically, we will take a region in the plane and rotate it about an axis to produce a solid. (You can visualize pottery being made.)

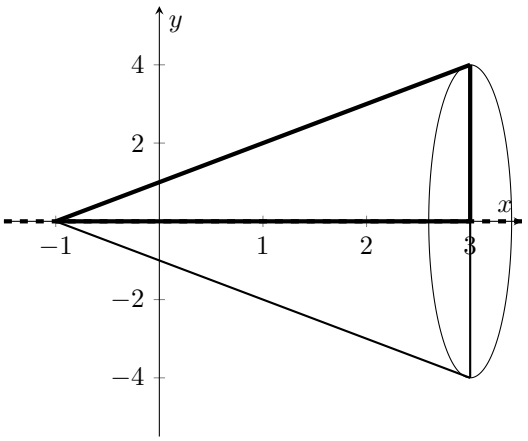
The inputs are the region in the plane and the axis of rotation. You will be asked to find the volume.

This is exactly like what we did with cross sections except now the cross sections will all be circles (or washers). You must do the following:

- Draw the region in the xy plane.
- Draw the axis of rotation.
- Draw the region reflected (rotated) about the axis of rotation.
- Sketch lightly some circles to indicate rotation (sometimes I omit this but it is important to practice it for a while).
- Identify the direction to slice (will the integral be dx or dy?)
- Identify the inner and outer radii
- Set up the integral and integrate:

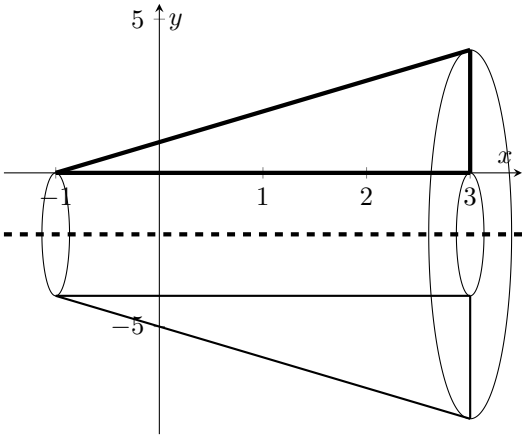
$$V = \int_a^b \left(\pi r_0^2 - \pi r_i^2\right)\,dx$$

Here's an example. Let R be the region in the plane bounded by $y = x + 1$, the x axis and $x = 3$. Find the volume of this rotated about the x axis.



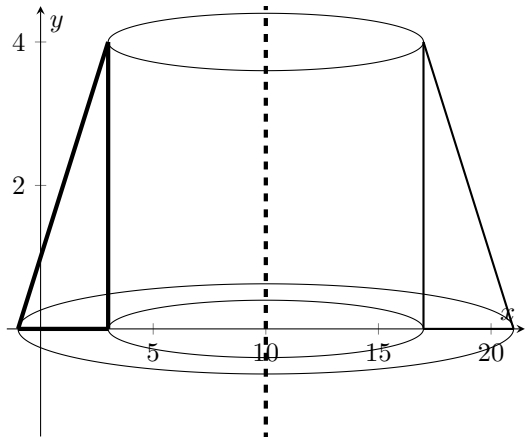
$$V = \int_{-1}^{3} \pi y^2 \, dx = \int_{-1}^{3} \pi (x+1)^2 \, dx = \frac{64\pi}{3}$$

What if you rotate about the axis $y = -2$. In this case you will have an inner and outer radius.



$$V = \int_{-1}^{3} \pi (r_o^2 - r_i^2) \, dx = \int_{-1}^{3} \pi \left[(y_{\text{line}} + 2)^2 - (y_{\text{horz}} + 2)^2 \right] \, dx$$
$$= \int_{-1}^{3} \pi \left[(x+1+2)^2 - (2)^2 \right] \, dx = \int_{-1}^{3} \pi \left[(x+3)^2 - (2)^2 \right] \, dx = \frac{160\pi}{3}$$

What if you rotate about the axis $x = 10$.



$$V = \int_0^4 \pi(r_o^2 - r_i^2)\, dy = \int_0^4 \pi\left[(10 - x_{\text{line}})^2 - (7)^2\right]\, dy$$
$$= \int_0^4 \pi\left[(10 - (y-1))^2 - (7)^2\right]\, dx$$
$$= \int_0^4 \pi\left[(11 - y)^2 - (7)^2\right]\, dx = \frac{400\pi}{3}$$

10. Let R_1 be the region from Problem 2.
    (a) Draw this region rotated around the y-axis.
    (b) Set up an integral representing this volume.
    Solution: $V = \int_0^4 \pi(\sqrt{y})^2\, dy = 8\pi$
    (c) Find the volume.

11. Let R_1 be the region from Problem 2.
    (a) Draw this region rotated around the x-axis.
    (b) Set up an integral representing this volume.
    Solution: $V = \int_0^2 \pi(4^2 - (x^2)^2)\, dx = \frac{128\pi}{5}$
    (c) Find the volume.

12. Let R_1 be the region from Problem 2.
    (a) Draw this region rotated around the line $x = 3$.
    (b) Set up an integral representing this volume.
    Solution: $V = \int_0^4 \pi(3^2 - (3 - \sqrt(y))^2)\, dy = 24\pi$
    (c) Find the volume.

13. Let R_1 be the region from Problem 2.
    (a) Draw this region rotated around the line $y = -2$.
    (b) Set up an integral representing this volume.
    Solution: $V = \int_0^2 \pi(6^2 - (2 + x^2)^2)\, dx = \frac{27704\pi}{15}$
    (c) Find the volume.

14. Let R_2 be the region from Problem 3.
    (a) Draw this region rotated around the y-axis.
    (b) Set up an integral representing this volume.
    Solution: $V = \int_0^4 \pi((\sqrt{y})^2 - (y/2)^2)\, dy = \frac{8\pi}{3}$
    (c) Find the volume.

15. Let R_2 be the region from Problem 3.
    (a) Draw this region rotated around the x-axis.
    (b) Set up an integral representing this volume.
    Solution: $V = \int_0^2 \pi((2x)^2 - (x^2)^2)\, dx = \frac{64\pi}{15}$
    (c) Find the volume.

16. Let R_2 be the region from Problem 3.
    (a) Draw this region rotated around the line $x = 3$.
    (b) Set up an integral representing this volume.
    Solution: $V = \int_0^4 \pi((3 - y/2)^2 - (3 - sqrt(y))^2)\, dy = \frac{16\pi}{3}$

(c) Find the volume.

17. Let R_2 be the region from Problem 3.

(a) Draw this region rotated around the line $y = -2$.

(b) Set up an integral representing this volume.

Solution: $V = \int_0^2 \pi((2+2x)^2 - (2+x^2)^2)\, dx = \frac{48\pi}{5}$

(c) Find the volume.