Math 132: Discussion Session: Week 4

Directions: In groups of 3-4 students, work the problems on the following page. Below, list the members of your group and your answers to the specified questions. Turn **this paper** in at the end of class. You do not need to turn in the question page or your work.

Additional Instructions: It is okay if you do not completely finish all of the problems. Also, each group member should work through each problem, as similar problems may appear on the exam.

Scoring:

Correct answers	Grade
0–3	0%
4–6	80%
7–11	100%

Group Members:

5.5: The Substitution Rule.

(1) (a) $\displaystyle\int_0^3 x\sqrt{x^2+16}\,dx =$

(b) $\displaystyle\int (\cot x)\ln(\sin x)\,dx =$

(c) $\displaystyle\int \cot x\,dx =$

(d) $\displaystyle\int \frac{dx}{(1+\sqrt{x})^5} =$

(2) a and b equal:

6.1: Area between curves.

(1) The area is:

(2) The area is:

(3) The area is:

(4) The area is:

(5) The area is:

(6) The area is:

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5.5: The Substitution Rule.

(1) Compute the following integrals using the substitution rule:

(a) $\displaystyle\int_0^3 x\sqrt{x^2+16}\,dx$

Solution: The substitution $u = x^2 + 16$ seems reasonable. We compute that $du = 2x\,dx$. In this case, the bound $x = 0$ corresponds to $u = 16$, and the bound $x = 3$ corresponds to $u = 25$. We can then use substitution to compute the definite integral.

$$\begin{aligned}
\int_0^3 x\sqrt{x^2+16}\,dx &= \tfrac{1}{2}\int_0^3 \sqrt{x^2+16}\cdot 2x\,dx \\
&= \tfrac{1}{2}\int_{16}^{25} \sqrt{u}\,du \\
&= \tfrac{1}{2}\left(\tfrac{2}{3}u^{3/2}\right)\Big|_{x=16}^{25} \\
&= \tfrac{1}{3}\left(25^{3/2} - 16^{3/2}\right) \\
&= \tfrac{1}{3}\left(5^3 - 4^3\right) \\
&= \tfrac{1}{3}(5-4)\left(5^2 + 5\cdot 4 + 4^2\right) \\
&= \tfrac{1}{3}\cdot 1 \cdot 61 \\
&= \boxed{\frac{61}{3}}.
\end{aligned}$$

(b) $\displaystyle\int (\cot x)\ln(\sin x)\,dx$

Solution: We write

$$\int \cot x \ln(\sin x)\,dx = \int \frac{\cos x}{\sin x}\cdot \ln(\sin x)\,dx.$$

It's somewhat tricky to see the right substitution here, but it's reasonable to try $u = \sin x$, so $du = \cos x\,dx$. Thus, we have

$$\int \frac{\cos x}{\sin x}\cdot \ln(\sin x)\,dx = \int \frac{\ln(\sin x)}{\sin x}\cos x\,dx = \int \frac{\ln u}{u}\,du.$$

We don't immediately know how to do this integral, but it's for sure simpler than the one we started with, so we can try a second substitution. A good choice here is $v = \ln u$, so $dv = \frac{1}{u}\,du$. We compute

$$\begin{aligned}
\int \frac{\ln u}{u}\,du &= \int \ln u\left(\frac{1}{u}\,du\right) \\
&= \int v\,dv \\
&= \tfrac{1}{2}v^2 + C \\
&= \tfrac{1}{2}(\ln u)^2 + C \\
&= \boxed{\tfrac{1}{2}(\ln(\sin x))^2 + C.}
\end{aligned}$$

What this means is that if we had started with the subtitution $v = \ln(\sin x)$ from the beginning, we would have solved the problem with just one substitution. If you did that, nice work! But you can solve the problem equally well by making things simpler one step at a time.

(c) $\displaystyle\int \cot x\,dx$

Solution: Writing

$$\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx,$$

we see that $u = \sin x$ is a good substitution choice, because then we get $du = \cos x \, dx$, which we see above. We write

$$\begin{aligned} \int \frac{\cos x}{\sin x} \, dx &= \int \frac{1}{\sin x} (\cos x \, dx) \\ &= \frac{1}{u} \, du \\ &= \ln |u| + C \\ &= \boxed{\ln |\sin x| + C.} \end{aligned}$$

(d) $\displaystyle\int \frac{dx}{(1+\sqrt{x})^5}$

Solution: Here, $u = 1 + \sqrt{x}$ is a reasonable choice for substitution, and we have $du = \frac{1}{2\sqrt{x}} \, dx$. We compute

$$\int \frac{dx}{(1+\sqrt{x})^5} = \int \frac{1}{(1+\sqrt{x})^5} \cdot 2\sqrt{x} \cdot \frac{1}{2\sqrt{x}} \, dx.$$

We see that we'll have to somehow write $2\sqrt{x}$ in terms of u. Rearranging our equation for u, we see that $\sqrt{x} = u - 1$. Thus, we can continue by rewriting the above expression in terms of u.

$$\begin{aligned} \int \frac{1}{(1+\sqrt{x})^5} \cdot 2\sqrt{x} \cdot \frac{1}{2\sqrt{x}} \, dx &= \int \frac{1}{u^5} \cdot 2(u-1) \, du \\ &= 2 \int \left(u^{-4} - u^{-5}\right) \, du \\ &= 2\left(-\tfrac{1}{3}u^{-3} + \tfrac{1}{4}u^{-4}\right) + C \\ &= \frac{1}{2u^4} - \frac{2}{3u^3} + C \\ &= \boxed{\frac{1}{2\left(1+\sqrt{x}\right)^4} - \frac{2}{3\left(1+\sqrt{x}\right)^3} + C.} \end{aligned}$$

(2) Find numbers a and b such that

$$\int_a^b (u^2+1) \, du = \int_{-\pi/4}^{\pi/4} \sec^4\theta \, d\theta.$$

Hint: $\sec^2\theta = \tan^2\theta + 1$, so

$$\sec^4\theta \, d\theta = \left(\tan^2\theta + 1\right)\left(\sec^2\theta \, d\theta\right).$$

Solution: It looks like we want to use a substitution to convert the integral in terms of θ into the integral in terms of u. What subtitution should we use? Perhaps the hint will help. Comparing the expressions

$$(u^2+1) \, du,$$
$$\left(\tan^2\theta + 1\right)\left(\sec^2\theta \, d\theta\right),$$

we see that the u matches up with $\tan\theta$. If we try $u = \tan\theta$, we see that, indeed, $du = \sec^2\theta \, d\theta$.

At this point, we look at the bounds. The lower bound $\theta = -\pi/4$ corresponds to $u = \tan(-\pi/4) = -1$, and the upper bound $\theta = \pi/4$ corresponds to $u = \tan(\pi/4) = 1$. Thus, we compute that

$$\int_{-\pi/4}^{\pi/4} \sec^2\theta \, d\theta = \int_{-\pi/4}^{\pi/4} \left(\tan^2\theta + 1\right) \sec^2\theta \, d\theta = \int_{-1}^{1} (u^2+1) \, du.$$

Thus, the bounds are $\boxed{a = -1 \text{ and } b = 1.}$

6.1: Areas between Curves. Compute the areas of the regions bounded by the following curves:

(1) $y = e^x$, $y = e^{-x}$, $y = 3$

Solution: Looking at the graph, we find that $y = e^x$ and $y = e^{-x}$ intersect at $(x, y) = (0, 1)$. We can solve that $y = e^x$ intersects $y = 3$ at $x = \ln 3$. Likewise, $y = e^{-x}$ intersects $y = 3$ and $x = -\ln 3$. We see that the picture is symmetric about the y-axis, so it suffices to compute the area for $x > 0$ and the double the result to get the answer.

To compute the area on the right-hand side, we compute

$$\int_0^{\ln 3} (3 - e^x)\, dx = (3x - e^x)\big|_0^{\ln 3} = 3(\ln 3 - 0) - (e^{\ln 3} - e^0) = 3\ln 3 - (3 - 1) = 3\ln 3 - 2.$$

As discussed above, we must double the result to obtain our answer of $\boxed{6\ln 3 - 4.}$

(2) $y = \cos x$, $y = \cos(2x)$, $x = 0$, $x = \frac{2\pi}{3}$

Solution: Conveniently, $y = \cos x$ intersects $y = \cos(2x)$ at $x = 0$ and $x = 2\pi/3$. Indeed, plugging $x = 0$ into either expression gives $y = 1$, and plugging in $x = 2\pi/3$ into either expression gives $y = -\frac{1}{2}$.

We can, of course, also find all of the intersection points by solving the equation $\cos x = \cos(2x) = 2\cos^2 x - 1$. Solving the quadratic equation $2\cos^2 x - \cos x - 1 = 0$ we find that $\cos x = 1$ or $\cos x = -\frac{1}{2}$. In the range $0 \le x \le \frac{2\pi}{3}$, the equation $\cos x = 1$ is only satisfied when $x = 0$, and $\cos x = -\frac{1}{2}$ is only satisfied when $x = 2\pi/3$, as we saw previously.

The graph makes it clear which function is larger, though if we pick the wrong one we'll be able to see that we made a mistake when we get a negative area. We can use a quick substitution, or simply guessing and correcting, to find that the antiderivative of $\cos 2x$ is $\frac{1}{2}\sin 2x$. We compute that the area is

$$\int_0^{2\pi/3} (\cos x - \cos 2x)\, dx = \left(\sin x - \tfrac{1}{2}\sin 2x\right)\Big|_0^{2\pi/3} = \left(\frac{\sqrt{3}}{2} - 0\right) - \frac{1}{2}\left(-\frac{\sqrt{3}}{2} - 0\right) = \boxed{\frac{3\sqrt{3}}{4}.}$$

(3) $y = |x|$, $y = 2 - |x|$

Solution: Looking at the graph, we see that the region is a square. One of the vertices of the square is $(x, y) = (0, 0)$. To compute the coordinates of the vertex in the first quadrant, we see that $x > 0$, so $|x| = x$, and we find that $y = x$ intersects $y = 2 - x$ when $(x, y) = (1, 1)$.

Using the distance formula, we see that the side length of the square is $\sqrt{2}$. Thus, the area of the square is $(\sqrt{2})^2 = \boxed{2}$.

(4) $x + y = 4$, $x - y = 0$, $3x + y = 4$

Solution: Looking at the graph, we see that we'd like to compute some intersection points. The line $x + y = 4$ and $3x + y = 4$ intersect at $(x, y) = (0, 4)$. The lines $x - y = 0$ and $3x + y = 4$ intersect at $(x, y) = (1, 1)$. The lines $x + y = 4$ and $x - y = -0$ intersect at $(x, y) = (2, 2)$.

The line $x + y = 4$ has slope -1 and the line $x - y = 0$ has slopoe 1, so the two lines are perpendicular. Thus, the region we are looking at is a right triangle. Using the distance formula, we see that one of its legs has length $\sqrt{2}$, and the other leg has length $2\sqrt{2}$. Thus, the area of the region is $\frac{1}{2}\left(\sqrt{2}\right)\left(2\sqrt{2}\right) = \boxed{2}$.

Alternatively, looking at the graph, we see that we can compute the integral of the region by splitting it up into the ranges $0 \le x \le 1$ and $1 \le x \le 2$. Looking at which functions bound the triangle in those ranges, we see that, to find the area we need to compute the integrals

$$\int_0^1 ((4 - x) - (4 - 3x))\, dx + \int_1^2 ((4 - x) - x)\, dx = \int_0^1 2x\, dx + \int_1^2 (4 - 2x)$$
$$= x^2\big|_0^1 + 4x\big|_1^2 - x^2\big|_1^2 = 1 + 4 - (4 - 1) = 2,$$

as before.

(5) $x = \sin y$ and $x = 1 - \cos y$, for $0 \le y \le \pi$

Solution: The first thing we notice here is that the variables x and y have swapped roles. To make things simpler, in the figure, I have also swapped the labels of the axes.

Looking at the figure, this problem has the complication that the desired region has two pieces, which we need to address separately.

It appears from the graph that the two curves intersect at $(y, x) = (0, 0)$ and at $(y, x) = (\frac{\pi}{2}, 1)$. We can plug these in to check that these are valid intersection points.

If one is unconvinced by the graph, there are several ways to justify that there are no other intersection points. From an algebraic standpoint, the easiest is to notice that the curves intersect when $\sin y + \cos y = 1$. As a consequence $1 = (\sin y + \cos y)^2 = \sin^2 y + \cos^2 y + 2 \sin y \cos y = 1 + \sin(2y)$. Solving, we find that $\sin(2y) = 0$. Thus, the intersections can only happen at $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and so forth. In the range $0 \leq y \leq \pi$, there curves intersect at $y = 0$ and $y = \frac{\pi}{2}$, but not at $y = \pi$. The reason for that is that, at $y = \pi$, we still have $(\sin y + \cos y)^2 = 1$, but $\sin y + \cos y = -1$, not $+1$ like we needed.

In any case, the next step is to use an integral to compute that the area is

$$\int_0^{\pi/2} (\sin y - (1 - \cos y))\, dy + \int_{\pi/2}^{\pi} ((1 - \cos y) - \sin y)\, dy$$
$$= (-\cos y - y + \sin y)|_0^{\pi/2} + (y - \sin y + \cos y)|_{\pi/2}^{\pi}$$
$$= \left(1 - \tfrac{\pi}{2} + 1\right) + \left(\tfrac{\pi}{2} + 1 - 1\right) = \boxed{2}.$$

(6) $x = y^3 - 4y$, $x + 3y = 0$

Solution: As in the previous problem, the roles of x and y are swapped, and, as before, we see from the graph that there are two regions. Howevever, we see that we have symmetry: Both $f(y) = y^3 - 4y$ and $f(y) = -3y$ are odd functions. Thus, the two regions have the same area, so we can compute the area of the region with $y \geq 0$ and then multiply the result by two to get our answer.

To find the intersection of $x = y^3 - 4y$ and $x = -3y$, we set $y^3 - 4y = -3y$ and compute that $0 = y^3 - y = y(y + 1)(y - 1)$. Thus, the two curves intersect at $y = -1, 0, 1$. Computing x-values, we find that the interesection points are at $(y, x) = (-1, 3)$, $(y, x) = (0, 0)$, and $(y, x) = (1, -3)$. Thus, to find the area on the bottom right, we compute the integral

$$\int_0^1 \left(-3y - (y^3 - 4y)\right)\, dy = \int_0^1 (-y^3 + y)\, dy = (-\tfrac{1}{4}y^4 + \tfrac{1}{2}y^2)|_0^1 = \frac{1}{4}.$$

As discussed above, we only found the area of the region on the right. To find the total area, we need to multiply this result by two to get $\boxed{\frac{1}{2}}$.

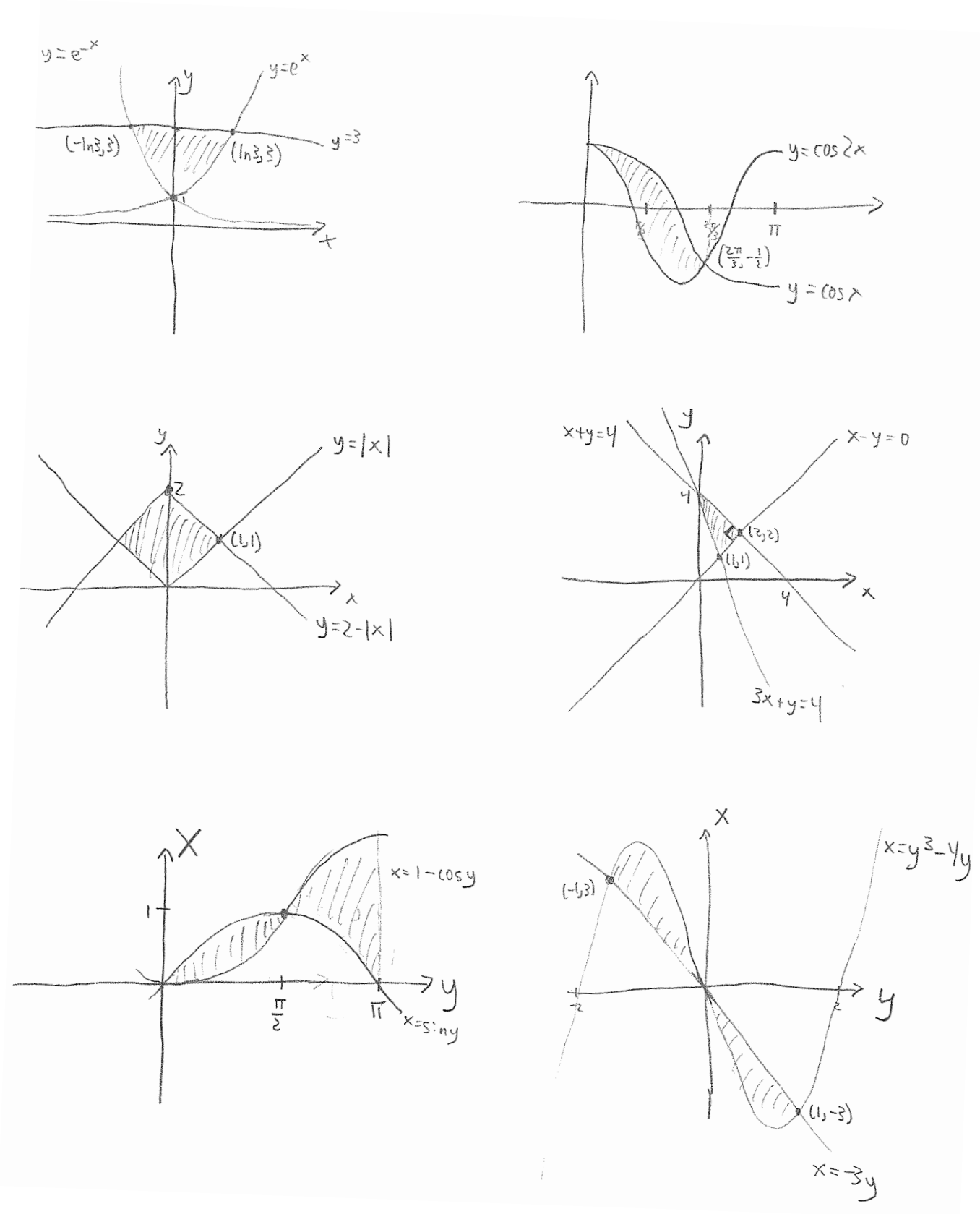

FIGURE 1. Area between curves.