Math 132 - September 18, 2017
Solutions

Chapter 6: Applications

Integration = Adding things up
Areas and Volumes

6.1: Areas

To find area between curves:

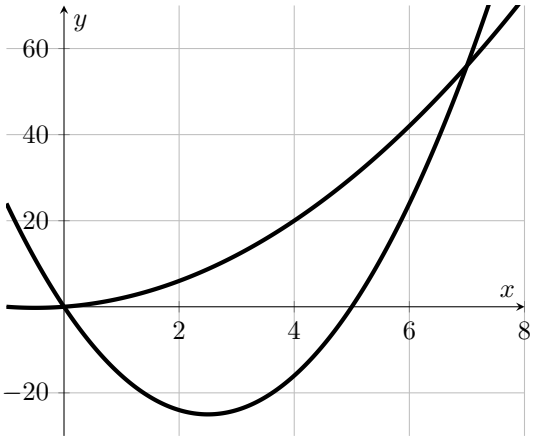
- Graph region, find intersections
- Decide if you are going to integrate dx or dy
- Area $= \int_a^b (\text{Big} - \text{Small})\, dx$

Warm-up Problems

1. Determine where the curves intersect.

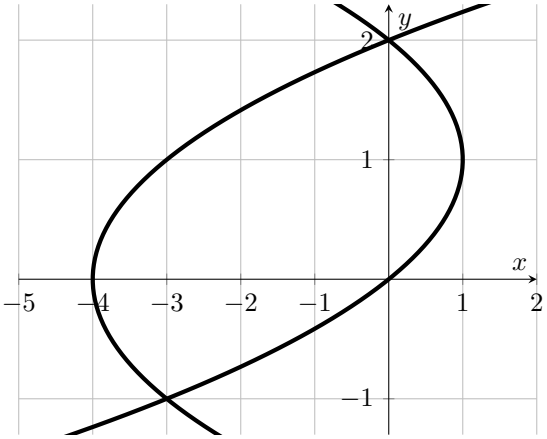
   (a) $y = x^2 + x$ and $y = 4x^2 - 20x$

   Solution: Solve to get $(0, 0)$ and $(7, 56)$



   (b) $x = y^2 - 4$ and $x = 2y - y^2$

   Solution: Solve $y^2 - 4 = 2y - y^2$ to get $(-3, -1)$ and $(0, 2)$.



Lecture Notes: To find the area between curves, it is essentially the same as area between a curve and the x-axis. Divide the domain, make rectangles, add up the rectangles. The widths of the rectangles is Δx, the heights of the rectangles is the difference between the curves, big minus small.

Make sure you draw the region. If you don't draw the region you will often not see the intersections correctly and will have a more difficult time working with dy vs dx.

Generally speaking, these area problems are pretty easy to set up correctly, IF the graph is drawn. They are even easy to set up about 80-90% of the time without a graph but the other times you'll likely get them wrong.
JUST DRAW THE GRAPHS!

Examples: find the areas of the first problems.

Class Problems

2. Find the area between the curves in Problem 1.

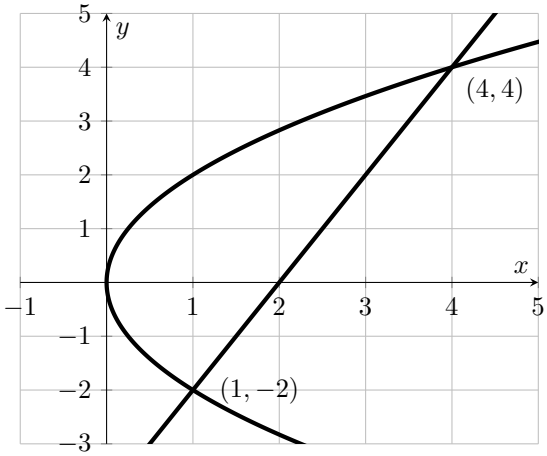
   Solution: For area between $y = x^2 + x$ and $y = 4x^2 - 20x$:

   $$A = \int_0^7 (x^2 - x) - (4x^2 - 20x)\, dx = \frac{245}{2}$$

   Solution: For the area between $x = y^2 - 4$ and $x = 2y - y^2$, the key point is that you want to integrate with respect to y and not x.
   $\int_{-1}^{2}(x_{\text{big}} - x_{\text{small}})\, dy = \int_{-1}^{2}(2y - y^2) - (y^2 - 4)\, dy = 9$
   This can also be done with a dx integral, but it isn't pretty. You have to solve for y, which is easy for one of the functions but a bit messy for the other (use quadratic formula):

   $$\begin{aligned} A =& \int_{-4}^{-3} \sqrt{x+4} - \left(-\sqrt{x+4}\right)\, dx + \int_{-3}^{0} \sqrt{x+4} - \left(1 - \sqrt{1-x}\right)\, dx + \int_{0}^{1} \left(1 + \sqrt{1-x}\right) - \left(1 - \sqrt{1-x}\right)\, dx \\ =& \frac{4}{3} + \frac{19}{3} + \frac{4}{3} = 9 \end{aligned}$$

3. **Clicker** Given the graphs of the functions below, find integral(s) that represent the area between $y^2 = 4x$ and $y = 2x - 4$.



   (a) $\displaystyle\int_{-2}^{4} \left(\frac{y+4}{2} - \frac{y^2}{4}\right)\, dy$ **Correct**

   (b) $\displaystyle\int_0^1 2\sqrt{4x}\, dx + \int_1^4 \left(\sqrt{4x} - (2x-4)\right)\, dx$ **Correct**

   (c) $\displaystyle\int_0^4 \sqrt{4x} - (2x-4)\, dx$

   (d) $\displaystyle\int_{-2}^{4} (4y^2 - 2x + 4)\, dy$

   (e) $\displaystyle\int_0^4 \left(\frac{y^2}{4} - (2x-4)\right)\, dx$

   Solution: Points of intersection are $(1, -2)$ and $(4, 4)$. The trick is the dx integral.

4. **Clicker** Find the area between $y = \sin x$ and $y = \cos x$ between $x = 0$ and $x = 2\pi$.

(a) 0

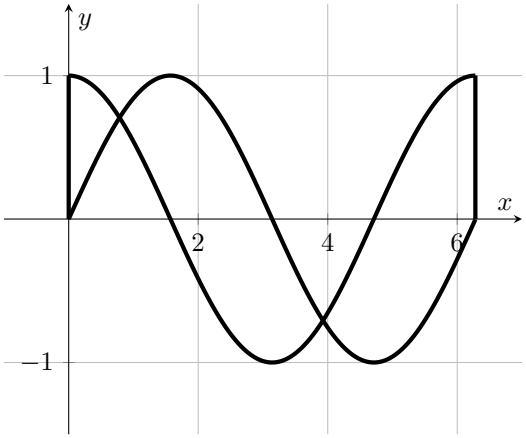
(b) 2

(c) $4\sqrt{2}$

(d) 4

(e) $-\infty$

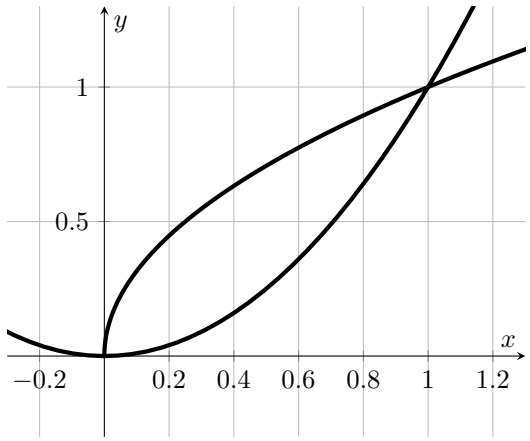
Solution:



$$A = \int_0^{\pi/4} (\cos x - \sin x)\ dx + \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\ dx + \int_{5\pi/4}^{2\pi} (\cos x - \sin x)\ dx$$
$$= (\sqrt{2} - 1) + (2\sqrt{2}) + (\sqrt{2} + 1) = 4\sqrt{2}$$

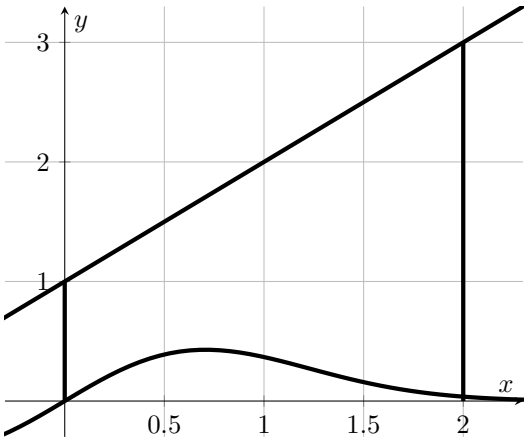
5. Find the area between $y = x^2$ and $y = \sqrt{x}$.
(Do this in two ways, as a dx integral and as a dy integral.)

Solution: $\frac{1}{3}$



6. Find the area between $y = xe^{-x^2}$, $y = x + 1$, $x = 2$ and the y axis.

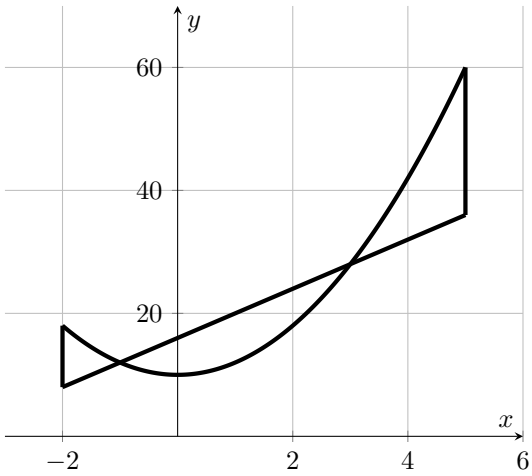
Solution: $A = \int_0^2 (x + 1 - xe^{-x^2})\ dx = \frac{7}{2} + \frac{e^{-4}}{2}$



7. Find the area between $y = 2x^2 + 10$, $y = 4x + 16$, $x = -2$ and $x = 5$.

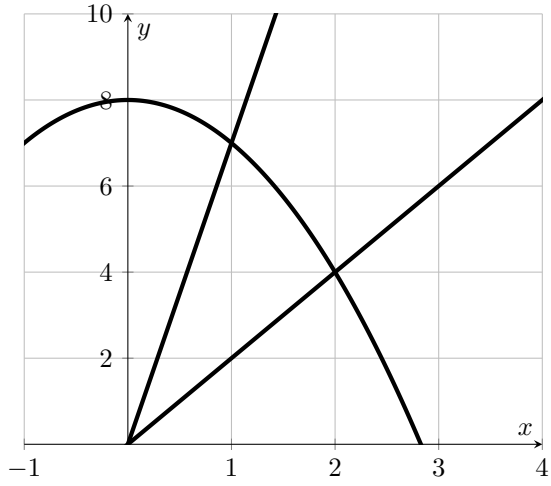
   Solution: This is important to break up (graph it and see why!!)

$$A = \int_{-2}^{-1} (2x^2+10) - (4x+16)\, dx + \int_{-1}^{3} (4x+16) - (2x^2+10)\, dx + \int_{3}^{5} (2x^2+10) - (4x+16)\, dx = \frac{14}{3} + \frac{64}{3} + \frac{64}{3} = \frac{142}{3}$$



8. Find the area enclosed by the graphs of $y = 8 - x^2$, $y = 7x$ and $y = 2x$ in the first quadrant.

   Solution:



$$A = \int_0^1 7x - 2x\, dx + \int_1^2 (8 - x^2) - 2x\, dx = \frac{31}{6}$$

or, you can do it dy:

$$A = \int_0^4 (y/2 - y/7)\ dy + \int_0^4 \left(\sqrt{8-y} - y/7\right)\ dy$$