Math 132 - September 6, 2017
Solutions

5.3: FTC

- FTC Part 1

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

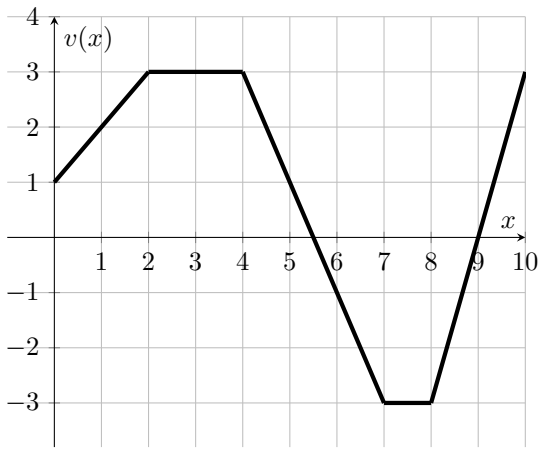
- FTC Part 2

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Warm-up Problems

1. **Clicker**

Use the graph of velocity of a car, $v(x)$, given above, where x is in seconds and $v(x)$ is in feet per second. Let $s(x) = \int_0^x v(t)\,du$ and find $s(10)$.
(What does $s(10)$ mean physically?)

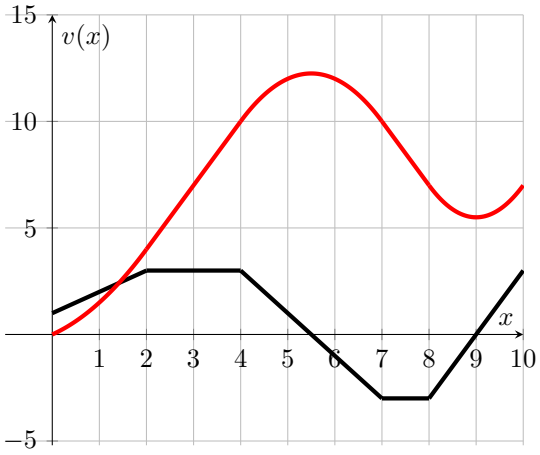


(a) -10
(b) 0
(c) 7 **Correct**
(d) 20 **Solution:** 20.5 is the "total distance traveled" and not the total displacment
(e) 100

Solution: $s(10)$ is the total displacement of object from time 0 to time 10.

2. Let $s(x) = \int_0^x v(t)\,dt$. Draw a graph of $s(x)$.

Solution:



Note how $s'(x) = v(x)$.

Lecture Notes: We are going to investigate functions of the form

$$g(x) = \int_a^x f(t)\,dt$$

where a is a constant.

Note the need for the variable t.

This is essentially the function in Problem 1. These functions can come up often so it is good to understand them.

Given such a $g(x)$, what is $g'(x)$:

$$g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt$$

For small h and continuous f, $f(t)$ is nearly constant. Thus the integral is approximately equal to $f(x)\cdot h$ and we have:

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt \\ &= f(x) \end{aligned}$$

This make sense if you think about the idea of rate of change. $g'(x)$ measures how much $g(x)$ is changing when x changes.

Another way to write this is

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

3. Chain Rule Practice. Remember that $[F(g(x))]' = F'(g(x))\,g'(x)$
Suppose you have a function $F(x)$ and you know $F'(x) = \cos(x^5+1)$. Find the following (hint: identify $g(x)$ in each problem and apply the chain rule).
   (a) $[F(x)]' = \cos(x^5+1)\cdot 1$
   (b) $\left[F(x^3)\right]' = \cos((x^3)^5+1)\cdot 3x^2$
   (c) $[F(\sin x)]' = \cos((\sin x)^5+1)\cdot \cos x$
   (d) $[F(\ln x)]' = \cos((\ln x)^5+1)\cdot \frac{1}{x}$

Class Problems

4. Find a function with the given derivative:
   (a) $f'(x) = \sin(x^2), \qquad f(x) = \int_0^x \sin(t^2)\,dt$
   (b) $f'(x) = e^{x^2}, \qquad f(x) = \int_{-2}^x e^{t^2}\,dt$
   (c) $f'(x) = e^{\cos x}, \qquad f(x) = \int_1^x e^{\cos x}\,dt$

5. Find the derivatives (compare to Problem 3):
   (a) $\frac{d}{dx}\int_0^x \cos(t^5+1)\,dt = \cos(x^5+1)\cdot 1$
   (b) $\frac{d}{dx}\int_0^{x^3} \cos(t^5+1)\,dt = \cos((x^3)^5+1)\cdot 3x^2$
   (c) $\frac{d}{dx}\int_0^{\sin x} \cos(t^5+1)\,dt = \cos((\sin x)^5+1)\cdot \cos x$
   (d) $\frac{d}{dx}\int_0^{\ln x} \cos(t^5+1)\,dt = \cos((\ln x)^5+1)\cdot \frac{1}{x}$

6. **Clicker** Find the derivative

$$\frac{d}{dx}\int_x^{x^2} \sin(t^2)\,dt$$

   (a) $\sin(x^2)\cdot 2x$

(b) $\sin(x^4) \cdot 2x$

(c) $\sin(x^2) - \sin(x)$

(d) $\sin(x^4) \cdot 2x - \sin(x^2)$

(e) There is no derivative because this problem self destructs

Solution: The key here is that you have to not only use the chain rule but split up the integral so the FTC part 1 applies:

$$\begin{aligned}
\frac{d}{dx}\int_x^{x^2} \sin(t^2)\,dt &= \frac{d}{dx}\left[\int_x^0 \sin(t^2)\,dt + \int_0^{x^2} \sin(t^2)\,dt\right] \\
&= \frac{d}{dx}\left[\int_0^{x^2} \sin(t^2)\,dt - \int_0^x \sin(t^2)\,dt\right] \\
&= \frac{d}{dx}\int_0^{x^2} \sin(t^2)\,dt - \frac{d}{dx}\int_0^x \sin(t^2)\,dt \\
&= \sin(x^4)(2x) - \sin(x^2)
\end{aligned}$$

Lecture Notes: The FTC part 1 essentially says that the derivative of an integral is equal to the function. What about the integral of a derivative? This turns out to be very useful: FTC, part 2.

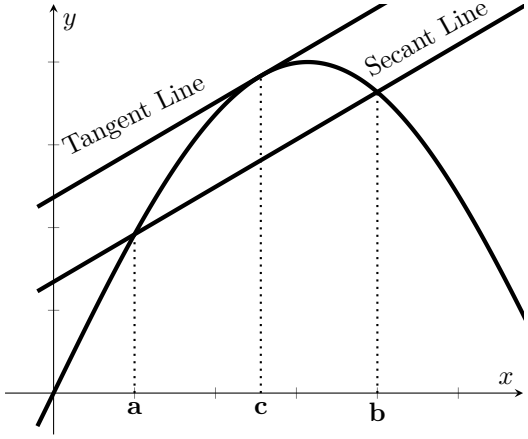
Here is the set up. We have a function $F(x)$ and its derivative $F'(x)$. We want to compute

$$\int_a^b F'(x)\,dx$$

We do this by doing a Riemann Sum with sample points given by the MVT.

7. **Clicker**

Explain the picture and the Mean Value Theorem (MVT)



(a) Given $f(x)$ and $a < b$, there is a c such that $f'(c) = \frac{f(b)-f(a)}{b-a}$ **Correct**

(b) Given a function there is always a tangent line parallel to a secant line. **Correct**

(c) Given a tangent line, there is a parallel secant line

(d) MVT? What are you talking about?

Lecture Notes: Back to the FTC part 2. We have a function $F(x)$ and its derivative $F'(x)$. We want to compute

$$\int_a^b F'(x)\,dx$$

We do this by doing a Riemann Sum with sample points given by the MVT.

What this means is the following. We compute a Riemann Sum and take the limit:

$$R_n = \sum_{i=1}^{n} F'(x_i^*)\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and x_i^* is chosen so that

$$F'(x_i^*) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} = \frac{F(x_i) - F(x_{i-1})}{\Delta x}$$

Plugging this in:

$$\begin{aligned} R_n &= \sum_{i=1}^{n} F'(x_i^*)\Delta x = \sum_{i=1}^{n} \left(\frac{F(x_i) - F(x_{i-1})}{\Delta x} \right) \Delta x \\ &= \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) = F(b) - F(a) \end{aligned}$$

Now take the limit as $n \to \infty$ and we get the result. (Notice that our cleverness mean we didn't have to take the limit.)

Note on the proof above: You are not required or expected to recreate this, but it is a nice proof of the FTC part 2. You are expected to be able to use the FTC part 2 and I believe that seeing why it is true is helpful.

The point of FTC part 2 is that to compute definite integrals, you just compute antiderivatives (indefinite integrals).

8. Find the following indefinite integrals using FTC part 2.

(a) $\displaystyle\int_1^9 -2 \, dx = -16$

(b) $\displaystyle\int_1^4 x \, dx = 15/2$

(c) $\displaystyle\int_{-1}^1 x^2 \, dx = 2/3$

(d) $\displaystyle\int_0^\pi \sin x \, dx = 2$

(e) $\displaystyle\int_1^{10} \frac{1}{x} \, dx = \ln 10$

(f) $\displaystyle\int_0^{\pi/4} \sec^2 x \, dx = 1$

(g) $\displaystyle\int_1^4 x \, dx = 15/2$