Math 132: Discussion Session: Week 2

Directions: In groups of 3-4 students, work the problems on the following page. Below, list the members of your group and your answers to the specified questions. Turn **this paper** in at the end of class. You do not need to turn in the question page or your work.

Additional Instructions: It is okay if you do not completely finish all of the problems. Also, each group member should work through each problem, as similar problems may appear on the exam.

Scoring:

Correct answers	Grade
0–3	0%
4–6	80%
7–14	100%

Group Members:

5.1–5.2: Riemann Sums and Limits of Riemann Sums.

(1) $L_3 =$

(2) $M_5 =$

(3) (a) $R_6 =$

(b) $R_6 = \sum_{i=1}^{6}$

(c) $R_{127} = \sum_{i=1}^{127}$

(d) $R_{127} =$

(e) $R_n = \sum_{i=1}^{n}$

(f) $R_n =$

(g) $\int_2^5 (5x^2 - 3)\,dx =$

(4) (a) $R_n = \sum_{i=1}^{n}$

(b) $R_n =$

(c) $\int_0^2 (x^3 - 2x)\,dx =$

5.2: Definite Integrals as Signed Area

(1) $\int_0^2 \sqrt{16 - 4x^2}dx =$

(2) $\int_{-2}^5 \Big(4 + x - 2|x|\Big)dx =$

Math 132 Discussion Session: Week 2

5.1–5.2: Riemann Sums and Limits of Riemann Sums. Useful formulas:

$$\sum_{i=1}^{n} 1 = n, \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

(1) Let $f(x) = \sin^2(x)$. Compute the Riemann sum L_3 with three intervals, sampling at left endpoints, on $[\frac{\pi}{6}, \frac{5\pi}{12}]$.

Solution: We have four intervals, so

$$\Delta x = \frac{\frac{5\pi}{12} - \frac{\pi}{6}}{3} = \frac{\pi}{12}.$$

Thus, our three sample points are

$$\begin{aligned} x_1^* &= \frac{\pi}{6}, \\ x_2^* &= \frac{\pi}{6} + \frac{\pi}{12} = \frac{\pi}{4}, \\ x_3^* &= \frac{\pi}{6} + 2\frac{\pi}{12} = \frac{\pi}{3}, \end{aligned}$$

We compute

$$\begin{aligned} L_3 &= \sum_{i=1}^{3} f(x_i^*)\Delta x \\ &= f(x_1^*)\Delta x + f(x_2)^*\Delta x + f(x_3^*)\Delta x \\ &= \left(\sin^2\tfrac{\pi}{6} + \sin^2\tfrac{\pi}{4} + \sin^2\tfrac{\pi}{3}\right) \cdot \frac{\pi}{12} \\ &= \left(\frac{1}{4} + \frac{1}{2} + \frac{3}{4}\right) \cdot \frac{\pi}{12} = \boxed{\frac{\pi}{8}} \approx \boxed{0.39}. \end{aligned}$$

(2) Let $f(x) = \ln(2x)$. Compute the Riemann sum M_5 with five intervals, sampling at midpoints, on $[1, 3]$.

Solution: We have five intervals, so the width of each rectangle is $\Delta x = \frac{3-1}{5} = \frac{2}{5}$. The left endpoint of the first rectangle is at 1, and we add half of the width to get to the midpoint, so the midpoint of the first rectangle is $x_1^* = 1 + \frac{1}{5} = \frac{6}{5}$. Adding $\frac{2}{5}$ each time, we find that the midpoints of the five intervals are

$$x_1^* = \frac{6}{5}, \qquad x_2^* = \frac{8}{5}, \qquad x_3^* = 2, \qquad x_4^* = \frac{12}{5}, \qquad x_5^* = \frac{14}{5}.$$

Then, we compute

$$\begin{aligned} M_5 &= \sum_{i=1}^{5} f(x_i^*)\Delta x \\ &= f(x_1^*)\Delta x + f(x_2)^*\Delta x + f(x_3^*)\Delta x + f(x_4^*)\Delta x + f(x_5^*)\Delta x \\ &= \boxed{\left(\ln\tfrac{12}{5} + \ln\tfrac{16}{5} + \ln 4 + \ln\tfrac{24}{5} + \ln\tfrac{28}{5}\right) \cdot \frac{2}{5}} \\ &= \boxed{\frac{2}{5}\ln\left(\frac{12 \cdot 16 \cdot 4 \cdot 24 \cdot 28}{5^4}\right)} \approx \boxed{2.69}. \end{aligned}$$

(3) Let $f(x) = 5x^2 - 3$ on $[2, 5]$.

(a) Compute the Riemann sum R_6 with six intervals, sampling at right endpoints.

Solution: We see that $\Delta x = \frac{5-2}{6} = \frac{1}{2}$. The right end-points of the intervals are at

$$x_1^* = \frac{5}{2}, \qquad x_2^* = 3, \qquad x_3^* = \frac{7}{2}, \qquad x_4^* = 4, \qquad x_5^* = \frac{9}{2}, \qquad x_6^* = 5.$$

Thus,

$$\begin{aligned}
R_6 &= \sum_{i=1}^{6} f(x_i^*)\Delta x \\
&= f(x_1^*)\Delta x + f(x_2)^*\Delta x + f(x_3^*)\Delta x + f(x_4^*)\Delta x + f(x_5^*)\Delta x + f(x_6^*)\Delta x \\
&= \Big(\Big(5\cdot\big(\tfrac{5}{2}\big)^2 - 3\Big) + \Big(5\cdot\big(\tfrac{6}{2}\big)^2 - 3\Big) + \Big(5\cdot\big(\tfrac{7}{2}\big)^2 - 3\Big) \\
&\qquad + \Big(5\cdot\big(\tfrac{8}{2}\big)^2 - 3\Big) + \Big(5\cdot\big(\tfrac{9}{2}\big)^2 - 3\Big) + \Big(5\cdot\big(\tfrac{10}{2}\big)^2 - 3\Big)\Big)\cdot\frac{1}{2} \\
&= \frac{5}{8}\left(5^2+6^2+7^2+8^2+9^2+10^2\right) - \frac{1}{2}(3+3+3+3+3+3) \\
&= \frac{5}{8}\left(\sum_{i=1}^{10} i^2 - \sum_{i=1}^{4} i^2\right) - 9 \\
&= \frac{5}{8}\left(\frac{10\cdot 11\cdot 21}{6} - \frac{4\cdot 5\cdot 9}{6}\right) - 9 = \boxed{\frac{1703}{8}} = \boxed{212.875}.
\end{aligned}$$

(b) Write down an expression for R_6 using $\sum_{i=1}^{6}$ notation. The only variable in your final answer should be i.

Solution: We know that $R_6 = \sum_{i=1}^{6} f(x_i^*)\Delta x$. Our final answer should only use the variable i, so we should write down x_i^* in terms of i. We see that the x_i^* form an arithmetic sequence with common difference $\frac{1}{2}$, so our formula for x_i^* should have a $\frac{1}{2}i$ in it. We also know that $x_1^* = \frac{5}{2}$, so to get the right formula, we must write

$$x_i^* = 2 + \tfrac{1}{2}i.$$

We check that, indeed, $x_1^* = 2 + \frac{1}{2} = \frac{5}{2}$ according to this formula. We then plug x_i^* into f, computing

$$\begin{aligned}
R_6 &= \sum_{i=1}^{6} f(x_i^*)\Delta x \\
&= \sum_{i=1}^{6} f\left(2 + \tfrac{1}{2}i\right)\cdot\frac{1}{2} \\
&= \boxed{\sum_{i=1}^{6}\left(5\left(2+\tfrac{1}{2}i\right)^2 - 3\right)\cdot\frac{1}{2}}.
\end{aligned}$$

(c) Write down an expression for R_{127} using $\sum_{i=1}^{127}$ notation. The only variable in your final answer should be i.

Solution: We first compute that $\Delta x = \frac{5-2}{127} = \frac{3}{127}$. The width of each rectangle is Δx, and we're sampling at right endpoints, so we know that our sample points are

$$x_1^* = 2 + \Delta x, \qquad x_2^* = 2 + 2\Delta x, \qquad x_3^* = 2 + 3\Delta x, \qquad x_4^* = 2 + 2\Delta x, \text{ etc.}$$

The pattern is clear:

$$x_i^* = 2 + i\Delta x = 2 + \tfrac{3}{127}i.$$

We are now ready to compute the Riemann sum.

$$\begin{aligned} R_{127} &= \sum_{i=1}^{127} f(x_i^*)\Delta x \\ &= \sum_{i=1}^{127} f\left(2+\tfrac{3}{127}i\right)\cdot\frac{3}{127}, \\ &= \boxed{\sum_{i=1}^{127}\left(5\left(2+\tfrac{3}{127}i\right)^2-3\right)\cdot\frac{3}{127}}. \end{aligned}$$

(d) Use the formulas at the top of the page to write down an expression for R_{127}. You don't need to simplify, but there should be no variables or summations in your final answer.

Solution: We need to split apart the sum into pieces that we can compute using the formulas at the top of the page. We compute that

$$\begin{aligned} R_{127} &= \sum_{i=1}^{127}\left(5\left(2+\tfrac{3}{127}i\right)^2-3\right)\cdot\frac{3}{127} \\ &= \frac{3}{127}\sum_{i=1}^{127}\left(5\left(4+\frac{12}{127}i+\frac{9}{127^2}i^2\right)-3\right) \\ &= \frac{3}{127}\left(\sum_{i=1}^{127}17+\sum_{i=1}^{127}\frac{60}{127}i+\sum_{i=1}^{127}\frac{45}{127^2}i^2\right) \\ &= \boxed{\frac{3}{127}\left(17\cdot 127+\frac{60}{127}\frac{127\cdot 128}{2}+\frac{45}{127^2}\frac{127\cdot 128\cdot(2\cdot 127+1)}{6}\right)} \\ &= \boxed{51+90\cdot\frac{128}{127}+\frac{45}{2}\cdot\frac{128(2\cdot 127+1)}{127^2}}. \end{aligned}$$

(e) Write down an expression for R_n using $\sum_{i=1}^{n}$ notation. The only variables in your final answer should be i and n.

Solution: We follow the same steps as before, except with n in place of 127. We see that $\Delta x = \frac{3}{n}$ and that

$$x_i^* = 2 + i\Delta x = 2 + \tfrac{3}{n}i.$$

Plugging that in, we see that

$$\begin{aligned} R_n &= \sum_{i=1}^{n} f(x_i^*)\Delta x \\ &= \sum_{i=1}^{n} f\left(2+\tfrac{3}{n}i\right)\cdot\frac{3}{n} \\ &= \boxed{\sum_{i=1}^{n}\left(5\left(2+\tfrac{3}{n}i\right)^2-3\right)\cdot\frac{3}{n}}. \end{aligned}$$

(f) Use the formulas at the top of the page to write down an expression for R_n. There should be no summations in your final answer, and the only variable in your answer should be n.

Solution: We follow the same steps as what we did with $n = 127$. We compute

$$\begin{aligned} R_n &= \sum_{i=1}^{n} \left(5\left(2+\tfrac{3}{n}i\right)^2 - 3\right) \cdot \frac{3}{n} \\ &= \frac{3}{n}\sum_{i=1}^{127} \left(5\left(4 + \frac{12}{n}i + \frac{9}{n^2}i^2\right) - 3\right) \\ &= \frac{3}{n}\left(\sum_{i=1}^{n} 17 + \sum_{i=1}^{n} \frac{60}{n}i + \sum_{i=1}^{n} \frac{45}{n^2}i^2\right) \\ &= \boxed{\frac{3}{n}\left(17 \cdot n + \frac{60}{n}\frac{n(n+1)}{2} + \frac{45}{n^2}\frac{n(n+1)(2n+1)}{6}\right)} \\ &= \boxed{51 + 90 \cdot \frac{n+1}{n} + \frac{45}{2} \cdot \frac{(n+1)(2n+1)}{n^2}}. \end{aligned}$$

(g) Compute the definite integral $\displaystyle\int_2^5 (5x^2 - 3)\,dx$ by computing $\lim_{n\to\infty} R_n$.

Solution: We can compute the limits of ratios of polynomials by looking at the leading coefficients, so

$$\lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \frac{n}{n} = 1,$$

and

$$\lim_{n\to\infty} \frac{(n+1)(2n+1)}{n^2} = \lim_{n\to\infty} \frac{2n^2}{n^2} = 2.$$

Thus,

$$\lim_{n\to\infty} R_n = 51 + 90 + \frac{45}{2} \cdot 2 = \boxed{186}.$$

(h) Compute the definite integral $\displaystyle\int_2^5 (5x^2 - 3)\,dx$ by computing the antiderivative of $5x^2 - 3$.

Solution: We see that

$$\int_2^5 (5x^2 - 3)\,dx = \left(\tfrac{5}{3}x^3 - 3x\right)\Big|_2^5 = \tfrac{5}{3}(5^3 - 2^3) - 3(5-2) = \boxed{186}.$$

(4) Let $f(x) = x^3 - 2x$ on $[0, 2]$.

(a) Write down an expression for R_n using $\sum_{i=1}^{n}$ notation. The only variables in your final answer should be i and n.

Solution: We start by computing that $\Delta x = \frac{2}{n}$. We see that $x_1^* = \Delta x$, $x_2^* = 2\Delta x$, $x_3^* = 3\Delta x$, and so forth. In general,

$$x_i^* = i\Delta x = \tfrac{2}{n}i.$$

We plug that into the formula for a Riemann sum.

$$\begin{aligned} R_n &= \sum_{i=1}^{n} f(x_i^*)\Delta x \\ &= \sum_{i=1}^{n} f\left(\frac{2}{n}i\right) \cdot \frac{2}{n} \\ &= \boxed{\sum_{i=1}^{n} \left(\left(\frac{2}{n}i\right)^3 - 2\left(\frac{2}{n}i\right)\right) \cdot \frac{2}{n}}. \end{aligned}$$

(b) Use the formulas at the top of the page to write down an expression for R_n. There should be no summations in your final answer, and the only variable in your answer should be n.

Solution: We simplify the above expression for R_n.

$$\begin{aligned} R_n &= \sum_{i=1}^{n} \left(\left(\frac{2}{n} i \right)^3 - 2 \left(\frac{2}{n} i \right) \right) \cdot \frac{2}{n} \\ &= \left(\frac{2}{n} \right)^4 \sum_{i=1}^{n} i^3 - \frac{8}{n^2} \sum_{i=1}^{n} i \\ &= \boxed{\frac{16}{n^4} \cdot \frac{n^2(n+1)^2}{4} - \frac{8}{n^2} \cdot \frac{n(n+1)}{2}} \\ &= \boxed{4 \cdot \frac{(n+1)^2}{n^2} - 4 \cdot \frac{n+1}{n}}. \end{aligned}$$

(c) Compute the definite integral $\int_0^2 (x^3 - 2x)\, dx$ by computing $\lim_{n\to\infty} R_n$.

Solution: As before, we know that $\lim_{n\to\infty} \frac{n+1}{n} = 1$, so $\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 = 1^2 = 1$. Thus,

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} 4 \cdot \frac{(n+1)^2}{n^2} - 4 \cdot \frac{n+1}{n} = 4 - 4 = \boxed{0}.$$

(d) Compute the definite integral $\int_0^2 (x^3 - 2x)\, dx$ by computing the antiderivative of $x^3 - 2x$.

Solution: We compute that

$$\int_0^2 (x^3 - 2x)\, dx = \left(\tfrac{1}{4}x^4 - x^2\right)\Big|_0^2 = \tfrac{1}{4} \cdot 2^4 - 2^2 = \boxed{0}.$$

5.2: Definite Integrals as Signed Area. Compute the following definite integrals by interpreting them as areas

(1) $\int_0^2 \sqrt{16 - 4x^2}\, dx$

Solution: The equation of the graph is $y^2 = 16 - 4x^2$ or

$$4x^2 + y^2 = 16.$$

This equation is the equation of an ellipse. Noting that $(2, 0)$ and $(0, 4)$ are points on this ellipse, we see that this ellipse is a circle of radius four that is compressed by a factor of two in the x direction. Thus, the area of the ellipse is half the area of the circle, that is, $\pi(4^2)/2 = 8\pi$.

Looking at the graph in the figure below, we see that the area under the curve is just a quarter of the ellipse, so the value of the integral is $8\pi/4 = \boxed{2\pi}$.

(2) $\int_{-2}^{5} \Big(4 + x - 2\,|x|\Big)\, dx$. Hint: Consider $x < 0$ and $x > 0$ separately to draw the graph of this function.

Solution: When $x < 0$, we have that $|x| = -x$, so $4 + x - 2\,|x| = 4 + 3x$. The graph is a straight line. When $x > 0$, we have that $|x| = x$, so $4 + x - 2\,|x| = 4 - x$. The graph is another straight line.

Using similar triangles or by solving the equations, we see that these lines have x-intercepts at $x = -\frac{4}{3}$ and $x = 4$. The triangle above the x-axis thus has a base of $4 + \frac{4}{3} = \frac{16}{3}$ and a height of 4, for an area of $\frac{32}{3}$. Meanwhile, the triangle below the x-axis on the left has a base of $\frac{2}{3}$ and a height of 2, for an area of $\frac{2}{3}$. The triangle below the x-axis on the right has a base of 1 and a height of 1 for an area of $\frac{1}{2}$. Thus,

$$\int_{-2}^{5} \Big(4 + x - 2\,|x|\Big)\, dx = \frac{32}{3} - \frac{2}{3} - \frac{1}{2} = \boxed{\frac{19}{2}}.$$

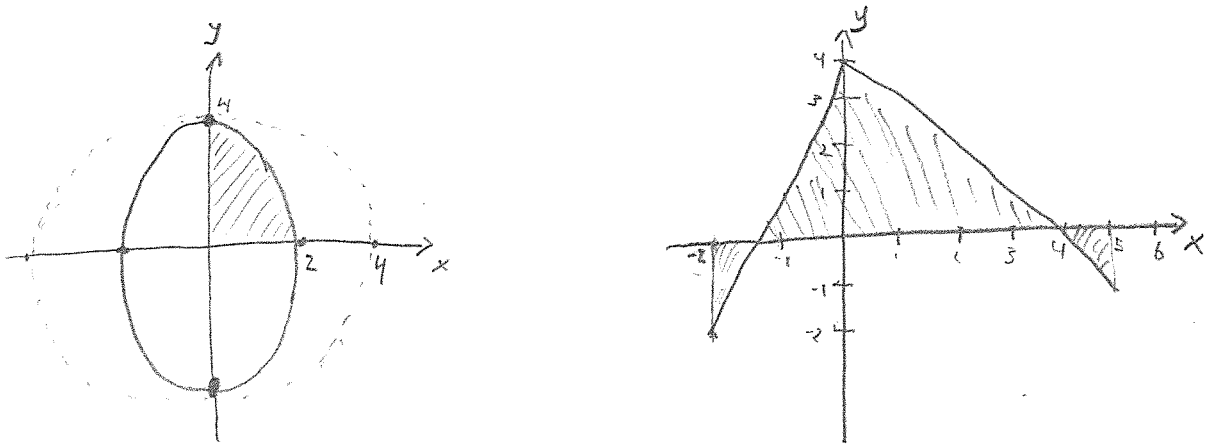

FIGURE 1. Definite integrals as signed area.