Developing Elementary Teachers' "Algebra Eyes and Ears"

Developing young children's capacity for algebraic thinking is an important strand of the recommendations in *Principles and Standards for School Mathematics* (NCTM 2000). The authors of this article, along with our colleagues at the National Center for Improving Student Learning and Achievement in Mathematics and Science, are studying ways to help elementary teachers bring to their students the kinds of rich and connected experiences in algebraic thinking that reform agendas such as *Principles and Standards* endorse. Although elementary teachers are essential to implementing this important change in the early grades, most elementary teachers have had little experience dealing with algebra since they were students in high school. They most likely experienced algebra as a collection of techniques for factoring and simplifying expressions, solving equations, and so on. They probably did not explore what the expressions or equations might mean, where they came from, and why anyone would manipulate them.

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Our Strategy: "Algebrafy" Elementary School Mathematics

Elementary teachers need their own experiences with a richer and more connected algebra and an understanding of how to build these opportunities for their students. Our strategy aims to help teachers learn to identify and create opportunities for algebraic thinking as part of their normal instruction and to use their own resources such as textbooks and supplementary materials. In particular, we help teachers focus on ways for students to generalize their mathematical thinking and express and justify their generalizations. Our

"algebrafication" strategy involves three types of teacher-based classroom change: algebrafying instructional materials, finding and supporting students' algebraic thinking, and creating classroom culture and teaching practices that promote algebraic thinking.

Algebrafying instructional materials

This term refers to generating algebraic activities from available instructional materials. Existing arithmetic activities and word problems are transformed from problems with a single numerical answer to opportunities for pattern building, conjecturing, generalizing, and justifying mathematical facts and relationships. Varying the given parameters of a problem is one simple yet powerful method. This is easily done with familiar tasks such as the Handshake Problem, which typically is posed as an arithmetic task with a single numerical answer: "How many handshakes will there be if each person in your group shakes the hand of every person once?" Students need simply to compute a sum, although they might first draw a picture or diagram to keep track of the people shaking hands. For example, they might record the initials of the person whose hand was shaken or draw a set of figures with lines between them indicating who has shaken hands. Tyler, a visually impaired student, used adaptive equipment to participate in the activity (see **fig. 1**).

To algebrafy the Handshake Problem, simply vary the number of people in the group (see **fig. 2**). How does this lead to algebraic thinking? By varying the number of people in the group, students are able to generate a set of data that has a mathematical pattern. A subtle but important part of this task is asking students about the number of handshakes for a group whose size is sufficiently large so that students cannot (or would not want to) draw or model the problem and write down a corresponding sum to compute. That is, to determine the number of handshakes in a group of 100, students cannot simply compute a sum; they must identify a pattern and think about how the amounts of handshakes for the various groups are related. (We refer to this as *algebraic use of numbers*.) From this, they can make a conjecture that describes what is true about the number of handshakes for a group of any size. When they support their conjec-

Figure 1

Tyler, a visually impaired student, used tactile tools to record data for the Handshake Problem. To add the numbers, he used a Brailler and an abacus.

Figure 2

"Algebrafying" the Handshake Problem

ture with sufficient evidence, the conjecture becomes a *generalization*, or a statement that describes a general truth that applies across multiple specific cases.

As another example, consider the following simple arithmetic task:

> I want to buy a T-shirt that costs $14. I have $8 already saved. How much more money do I need to earn to buy the shirt?

This problem can be extended to an algebraic task in several ways, such as by varying the cost of the shirt:

> Suppose the T-shirt costs $15, $16, $17, or $26. For each of these cases, write a number sentence that describes how much more money I need to buy the item. If *P* stands for the price of any T-shirt I want to buy, write a number sentence using *P* that describes how much more money I need to buy the T-shirt.

The underlying idea of this transformation is simple: Start with a typical numerical task and generalize it to a sequence of problems, each of which can be solved by an accompanying number sentence. The result is a sequence of number sentences that can be used to determine a generalized statement that captures the entire sequence in one statement; that is, $*P* – $8 = amount needed to earn. The intent is to get students to move beyond the arithmetic task and to think about the problem in general. Algebrafying tasks in this way is very flexible. To address a particular arithmetic reasoning agenda—multiplication, for example—you can vary the givens in a situation as follows:

> Jack wants to save up to buy a CD player that costs $35. He makes $5 per hour babysitting. How many hours will he need to work in order to buy the CD player?

Suppose that the price of the CD player varies; it is on sale for $30, or Jack decides to buy a more expensive one at $45. If you want to stress multiplying by 7, give Jack a raise to $7 per hour. Varying one of the problem givens in these ways creates a sequence of problems about which students can make a generalized statement such as "The number of hours Jack needs to work is the cost of the CD player divided by Jack's hourly wage." As applied mathematician Richard Hamming notes, "The purpose of computation is not numbers, it's insight" (Hamming 1962, epigraph).

Given the hectic pace of today's classrooms, a powerful benefit of using algebrafied arithmetic tasks is that students do many important things at once, including developing number sense, practicing number facts, and recognizing and building patterns to model situations. Algebrafied problems such as these can provide large amounts of computational practice in a context that intrigues students and avoids the mindlessness of numerical worksheets. Moreover, these tasks can be generated directly from existing materials.

Finding and supporting students' algebraic thinking

Teachers must focus on student thinking in order to develop their "algebra eyes and ears." They should ask simple questions such as the following:

- Tell me what you were thinking.
- Did you solve this a different way?
- How do you know this is true?
- Does this always work?

These questions not only reveal students' thinking but also prompt students to justify, explain, and build arguments—processes that lie at the heart of algebraic reasoning. Focusing on student thinking helps teachers see the depth, variety, and originality of students' arguments and often makes them think deeply about students' algebraic reasoning and the kinds of mathematical tasks that support it. Students demonstrate the beginnings of algebraic reasoning when they use particular numbers to argue a general case, or when they intend for a particular number to stand for any number and use gestures and vocal emphases to make their intent apparent. One third-grade teacher with whom we work described how students argued that "odd + even = odd":

> The only confusion came when Sarah said that odd plus even was odd and even plus odd was even. Stephen responded that that couldn't be. He used numbers in place of "odd" and "even" and said that using "odd" and "even" was the same as using letters. Sarah explained to the class, "I thought that all the time when odd is the first one it was supposed to be odd, and when even was first it was going to be even. [But then I realized that wasn't correct] because once you start turning them around, then it's the same thing. It doesn't make a difference."

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This vignette shows that Stephen recognized the terms "even" and "odd" as placeholders that he could replace with any even or odd numbers in order to reason about the sum. By explaining her thinking, Sarah was able to show how she used the generalization that addition is commutative to examine her own claim. When a child makes a generalization such as "Once you start turning them around, then it's the same thing. It doesn't make a difference," the teacher can push this algebraic thinking by asking the student why this might be true. The teacher can extend students' algebraic thinking further by changing the operation: "Do you think this is true when we subtract numbers or multiply numbers? Why or why not?"

When students attempt to use computation to solve tasks that they could better solve by reasoning algebraically, teachers can change the problem to use numbers beyond students' arithmetic capacity. For example, finding the number of handshakes in a group of one hundred people is far too cumbersome to solve by drawing the problem, recording every act of shaking hands, and computing the sum that results. The problem requires another technique. When students try to make a generalization about the sum of an even number and an odd number, they may focus on adding particular numbers instead of looking at properties of even and odd numbers. The teacher can change the numbers in question in order to shift the focus away from arithmetic to algebraic thinking. The following classroom episode, in which a teacher asked a student if the sum of 5 + 7 is even or odd, illustrates this:

Teacher. How did you get that?

Tory. I added 5 and 7, and then I looked over there [indicating a list of even and odd numbers on the wall] and saw that it was even.

Teacher. What about 45,678 plus 85,631? Odd or even?

Mary. Odd.

Teacher. Why?

Mary. Because 8 and 1 is even and odd, and even and odd is odd.

Students in this class had already discussed the result of adding even and odd numbers. By using sufficiently large even and odd numbers, the teacher eliminated the students' capacity to compute the sum, forcing them to think about the properties of the numbers. Mary knew that she needed to consider only the last digit of the two numbers and did not need to compute the actual sum. In fact, she did not even compute 8 + 1. She simply noted that 8 is even and 1 is odd, then used the generalization that "even + odd = odd" to get her solution.

Focusing on student thinking enables the teacher to see these types of situations as opportunities to engage students in making generalizations and in making those generalizations more explicit. This is an important aspect of having "algebra eyes and ears."

Creating a classroom culture and practices that promote algebraic reasoning

Promoting algebraic reasoning in the classroom involves incorporating conjecture, argumentation, and generalization in purposeful ways so that students consider arguments as ways to build reliable knowledge. It requires respecting and encouraging these activities as standard daily practice, not as occasional "enrichment" separate from the regular work of learning and practicing arithmetic. The following episode from Gina's first-grade classroom illustrates a classroom culture with these values. First, Gina took the arithmetic task "What is the greatest number of outfits you can make with 2 pairs of pants and 5 shirts?" and algebrafied it by varying the number of shirts or pants. Because this was a first-grade class, she created construction-paper cut-outs of different colors of shirts and pants that students could affix to the easel paper on which she recorded their comments. She made the following observations:

> I asked, "How many outfits can you make with this one red shirt, red pants, and blue pants?" Many children responded, "One." At this point, I needed to identify the meaning of *outfit*. Most students wanted to put like colors together. Each pair of partners then was instructed to pair red with red, and then blue with red. Then the action began. We recorded the information on chart paper. Another shirt was added to the two pairs of pants. Partners shared information and recorded it on the chart paper [see **fig. 3**]. I recognized that they needed to see this visually. After several activities and adding pants and shirts, the children were anxious to make predictions. After completing up to three shirts, the children began to see a pattern in the third column of outfits. Many responded, "We are counting by twos." Nathan recognized that 2 + 2 is a double and wondered if three shirts and three

Figure 3

The classroom chart used for the Outfit Problem

Shirts	Pants	Outfits
1	2	2
2	2	4
3	2	6

Figure 4

Sample in/out chart

IN	OUT
1	3
2	6
3	9

> pants would be six. He was able to figure out that there were nine outfits. We then used three shirts and four pants. Nathan predicted fourteen. Cory predicted ten and then said, "Let's do it to see." Matthew said, "Well, if one shirt and three pants is three outfits, then we just have to keep adding three."

Gina had created an environment that valued students modeling, exploring, arguing, predicting, conjecturing, and testing their ideas, as well as practicing computational skills. We promote the kind of mathematical talk described here, in which the teacher deliberately pushes students to generalize their thinking and build verbal and written arguments to justify their ideas. In this way, algebraic reasoning opportunities occur frequently and are viable. A classroom practice that supports conjecture, argumentation, generalization, and justification ensures that the two other parts of our algebrafication strategy—teachers' algebrafication of instructional resources and their focus on students' thinking—can be reproduced and sustained over the long term.

What Does an Algebrafied Practice Look Like?

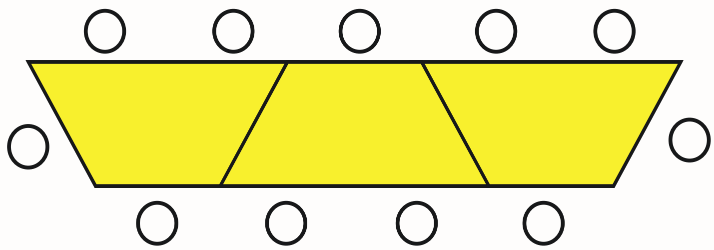
How do we know when a classroom practice has been algebrafied? In our work with teachers, we focused on the classroom practice of one third-grade teacher participant, Jan, who seemed to be algebrafying her classroom practice in a way that reflected our goals. We used a one-year case study of her classroom (Blanton and Kaput, forthcoming) to identify characteristics of Jan's practice that brought out the algebraic character of the mathematics. What follows is a discussion of these characteristics and classroom vignettes that illustrate them.

Algebraic themes are spiraled over periods of time

Jan did not treat an activity as an isolated task but instead revisited certain themes, such as relationships in data, properties of even and odd sums, or the use of letters to symbolize unknown or varying amounts, throughout the year in increasingly complex ways. For example, early in the year students used "in/out" charts to identify simple additive relationships in the "out" column. Students could determine that the "out" column in **figure 4** was increasing by 3 after looking for patterns in the column. During the year, this task progressed to looking at relationships between the "in" and "out" columns and identifying and symbolizing these functional relationships. Relationships between numbers were most clearly described later in the year as students solved the Trapezoid Problem, which required finding the number of people who could be seated at trapezoidal tables, modeled by adjoining various numbers of trapezoid pattern blocks (see **fig. 5**).

Figure 5

A three-desk configuration for the Trapezoid Problem



Students found the number of people who could sit at a table constructed of twelve trapezoid blocks by looking for a pattern in the data that described how many people could be seated at tables made of one, two, three, and four trapezoid blocks. They were focused on a pattern in a single data set, rather than a relationship between data sets. Jan described what occurred next:

> The strangest thing happened! I saw another pattern. I asked the class to look at the chart in another way. I wanted them to look at the relationship between the desks and the number of people. Find the rule. No luck. I then gave them the hint to see if there was a way to multiply and then add some numbers so it would always work. Jon suggested that we try to find a "secret message." After a few minutes, surprisingly, Anthony and Alicia started to multiply the number of desks with different numbers starting with 1. The two children found the rule of multiplying by 3 and then adding 2. We tried many examples from the chart and it worked each time. We even tried some big numbers like 100. We then tried to make a secret message. Andrew said that the 3 stays the same, so use a *d* for desk. This is what he came up with: $3(d) + 2$ = number of peo-

ple. The students realized that the 3 came from the people who could sit "on the top and the bottom" and the 2 came from the 2 sides.

Jan was able to spot ways to exploit algebraic thinking. In this case, she prompted students to think about relationships between data and to symbolize that in a fairly sophisticated way. The activity brought together a number of themes that Jan had addressed throughout the year: generating and organizing data, looking for patterns in single data sets, looking for relationships between quantities, symbolizing relationships in mathematically efficient ways (moving beyond everyday language), testing conjectures, and justifying mathematical generalizations. Each of these is a process that should become a habit of mind for elementary students. This will happen only if these processes become a routine part of classroom instruction.

Algebraic conversations are incorporated in the classroom on a regular and often spontaneous basis

Jan was able to engage students naturally in conversations that required them to generalize their thinking. She spontaneously shifted the focus away from computation to algebraic thinking during a rather ordinary task. Moreover, Jan's reflection about the Trapezoid Problem—"The strangest thing happened!"—shows that finding a relationship between the number of desks and the corresponding number of people was a spontaneous decision. The remainder of the conversation, in which students developed what they described as a secret message—"$3(d) + 2$ = number of people"—shows the flexibility with which Jan could extend the opportunity for algebraic thinking.

Teachers can identify or create arithmetic tasks that can be algebrafied

One of our most important goals is that teachers will be able to create their own set of instructional materials by algebrafying their existing resource base, so they are not dependent on the materials that we provided. Our intent was not to add on to teachers' own materials but to give them new, algebraic ways to look at what was already available. Shortly after her class had solved the algebrafied Handshake Problem (an activity we provided), Jan developed an activity based on the song "The Twelve Days of Christmas," which her class was learning for a school production. The simple task was stated as follows:

> How many gifts did your true love receive on each day? How many total gifts did he or she receive on the first 2 days? The first 3 days? The first 4 days? If the song was titled "The Twenty-Five Days of Christmas," how many total gifts would your true love receive?

Jan's problem had the same pattern and structure as the Handshake Problem. She selected the Trapezoid Problem described earlier because she was inspired by her committee work to select new school furniture. Her creativity in integrating algebraic thinking into a musical activity for her class and her capacity to see overlooked resources in new ways (the Trapezoid Problem was part of her existing resource base) are significant. Both examples show how she was learning to spot opportunities for algebraic thinking in places that were not obvious; she was developing "algebra eyes and ears."

Teachers contribute to the development of a school-wide culture of learning mathematics

The transformation that took place in Jan's classroom was not confined to the classroom or to her. With the support of her principal, she began to share with her colleagues both formally and informally what she was learning. She wrote, "I am slowly getting my student teacher into using one math problem that can be extended [algebrafied]." The student teacher gave the following problem to the class:

> The Millerville School is holding a car wash. A group of students can wash 6 cars in 1 hour. Suppose 2 groups of students wash cars at the same time. How many total cars could the groups wash in 3 hours? How many hours would it take to wash 72 cars?

Creating time to incorporate activities that involve algebraic thinking was one of the constraints that Jan and other teachers faced. As the examples here illustrate, these activities typically not only extend a "short" arithmetic task but also require student-centered discussions so that conjecturing and justification will thrive. As a school leader of a district-wide literacy initiative, Jan has worked to integrate algebraic thinking into her school's literacy program. Embedding the ideas of algebraic thinking into another subject area not only freed instructional time but also made both literacy and algebraic thinking more relevant for teachers. As an outgrowth of this endeavor, Jan has taken a lead role in establishing Monthly Math, a school-based project in which teachers identify "back of the textbook" concepts that often are overlooked and create a set of activities across the grades in which the whole school can participate. The result has been a growing school-wide appreciation of the more complex kinds of mathematics of which students are capable. As one teacher described, Monthly Math "united the school." We might add that such activities are also beginning to "algebrafy" the school.

A practice that includes the components described in this article can emerge from our algebrafication strategy and resources. This practice can thrive independently of our intervention and have an impact on the broader school culture. As we found in Jan's classroom, it also can lead to greater student achievement on standardized tests (Kaput and Blanton 2001).

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