The $1,000,000 MathPickle Problems

The MathPickle is in the process of proposing 13 unsolved problems in mathematics (one for each grade, K-12). These are problems that can be understood by students in these grades. These problems are nicely explained, via video, at the MathPickle webpage, see [3].

The MathPickle, together with the Pacific Institute for the Mathematical Sciences is currently looking for sponsors to put up a $1,000,000 prize for each of the problems.

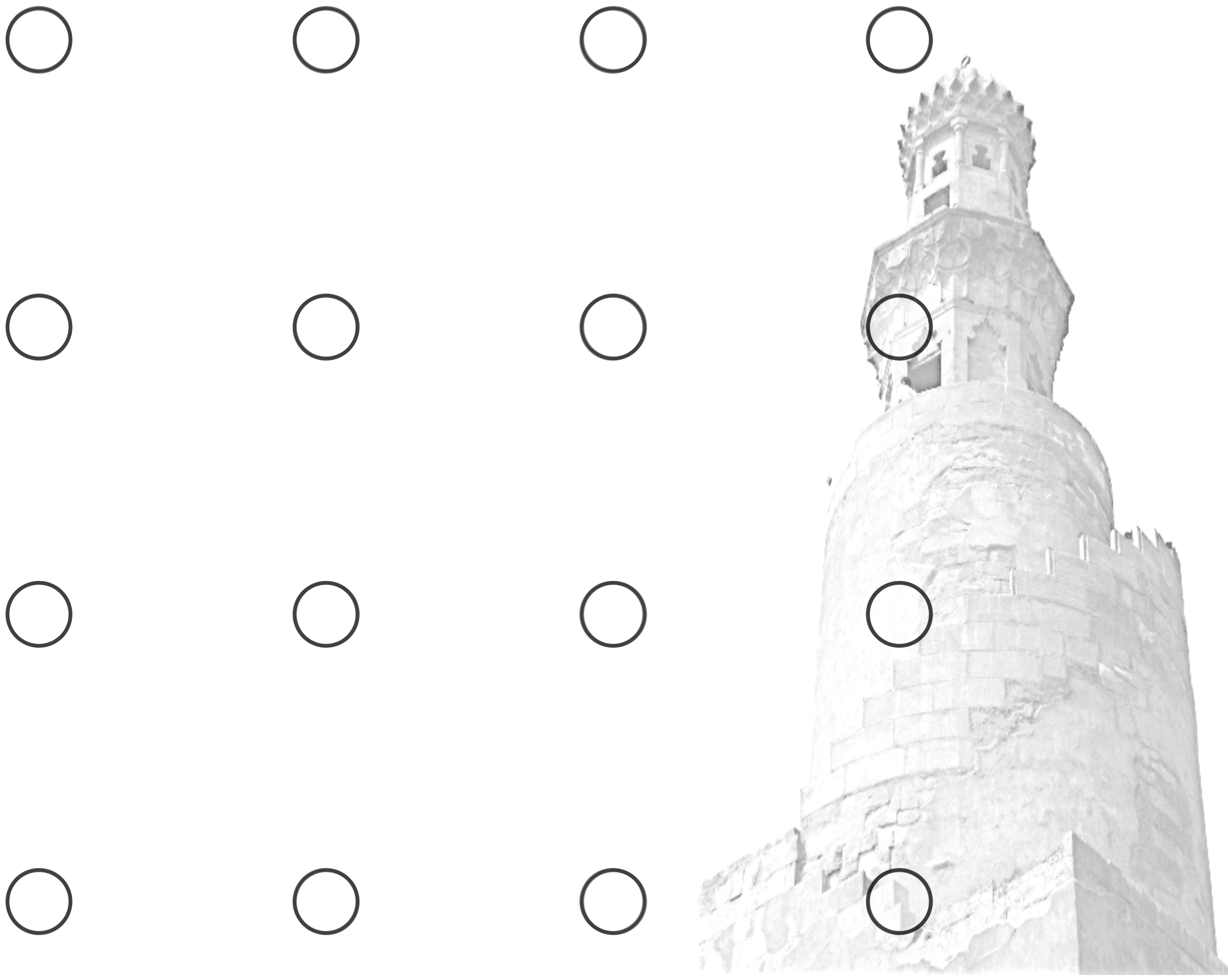
1 Grade 1: No-Three-In-Line

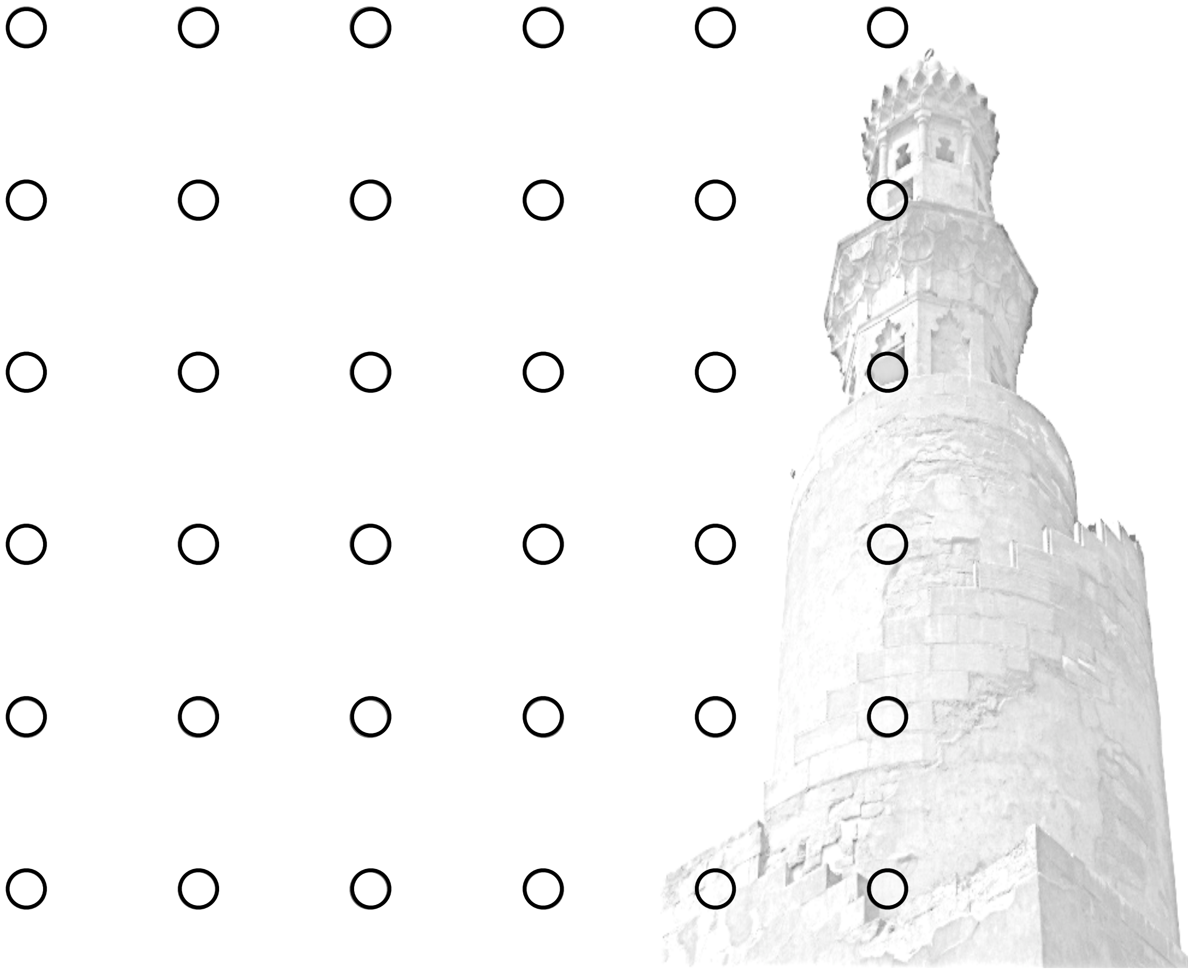
1.1 The Activity

Take a 4×4 grid and 8 manipulatives (or just color the dots). Place your 8 objects on the grid so that no three of the points lie in a line.

- How many different solutions can you find?
- More advanced: Try 10 points on a 5×5 grid. How many different solutions can you find?

1.2 Worksheets





1.3 The Mathematical Problem

This is called the *No-Three-In-A-Line Problem*, proposed by Henry Dudeney in 1917:

> Find the maximum number of points that can be places on the intersection points of an $n \times n$ grid, such that no three of the points lie in a line.

1.4 More Details and More Ideas

The wikipedia page on this is quite useful, see [2].

1. Solving the general case (what is a formula for the maximum number of points for any n) seems to be nearly impossible. But, working with specific cases is already quite difficult and there is still a lot unknown. For example, the largest n such that $2n$ points can be placed is currently $n = 52$.

2. For $n \leq 40$, it is known that $2n$ points can be placed with no three in a line. The number of solutions possible are:

n	2	3	4	5	6	7	8	9	10	11	12	12	14
Num Solutions	1	1	4	5	11	22	57	51	156	158	566	499	1366

3. (An upper bound) Try to convince yourself that for any n, at most $2n$ points can be marked (experiment with marking $2n + 1$ points and see what happens). This gives an upper bound–if you find a way to place $2n$ points then you have found the best possible solution.

4. (A lower bound) The above gives us an upper bound for the number n. Getting an estimate on least number is more difficult. It has been conjectured that for a given n, you can always place at least $\frac{\pi n}{\sqrt{3}}$ points and have no three collinear.

5. New Zealand Maths has a great collection of activities that start out easy and get more difficult. See [1] for more details.

2 Grade 2: Sum-Free Partitions

2.1 The Activity

Take the numbers: 1, 2, 3, 4, 5, 6, 7, 8. The object is to place these numbers, in any order, into two “magic hats”. The catch is that the numbers *explode* if the sum of any two numbers in a hat is equal to one of the other numbers.

- Add a third hat and see how many numbers you can fit in into three hats without any explosions.
- What is the best possible solution? What does best possible even mean?

2.2 The Mathematical Problem

This problem of Sum-Free Partitions was proposed by Issai Schur in 1916 and part of a field of mathematics called Ramsey Theory. Schur’s problem is

> Given a natural number k (the number of hats), find the largest possible n (the numbers to put in the hats), so that the number $1, 2, \ldots, n$ can be partitioned into k sum-free sets (placed into k hats, without exploding).

This largest possible n is called the Schur Number and is written $S(k)$ (where k is the number of hats).

2.3 More Details and More Ideas

1. If you start to investigate this on the Internet, be careful as the usual definition of Schur’s number allows for duplicating numbers in the sums. So, for example, you could not allow 4 and 8 in the same hat since $4 + 4 = 8$. The above problem with hats is actually to find *weak Schur numbers*.

2. Here are the (weak) Schur numbers that are known:

Number of Hats	2	3	4	5	6	7+
n	8	23	66	$w(5) \geq 196$	$w(6) \geq 572$	unknown

3. Known Schur numbers:

Number of Hats	2	3	4	5	6	7	8
n	4	13	44	$160 \leq S(5) \leq 315$	$S(6) \geq 536$	$S(7) \geq 1680$	unknown

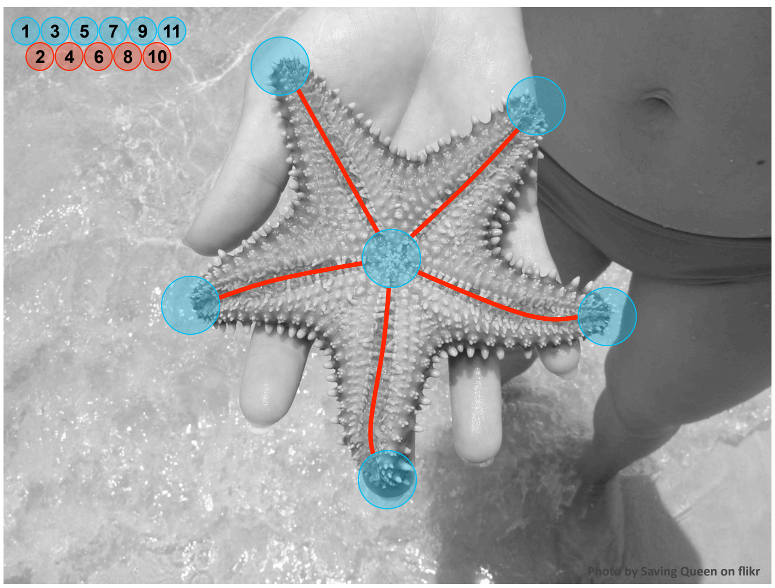
3 Grade 3:Graceful Tree Conjecture

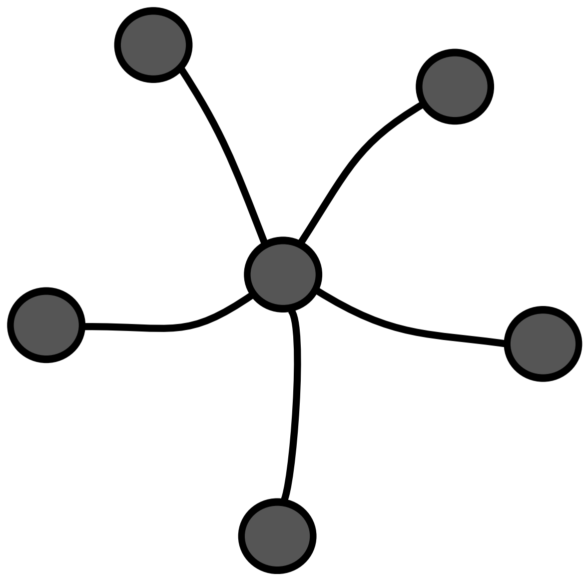
3.1 The Activity

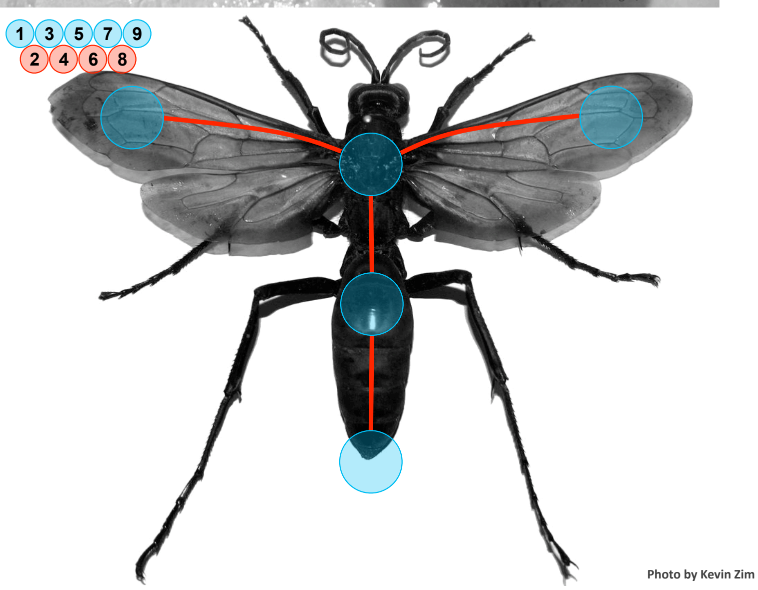
Place the odd consecutive integers $(1, 3, \ldots)$ on the circles (vertices). On each line connecting two odd integers (the edges), place their difference. The goal is to have all the even integers on the lines different.

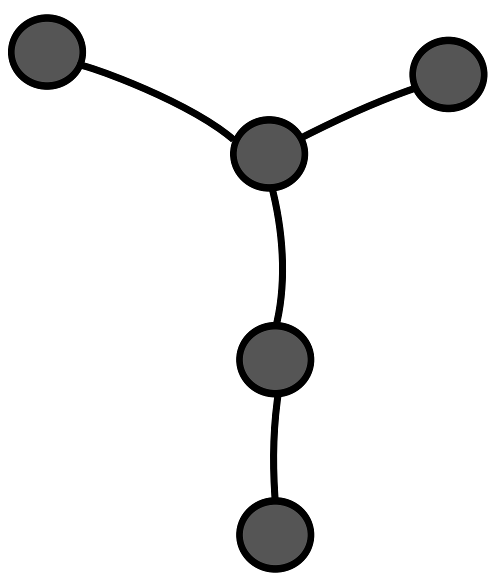
- Try to beat each of the animals in the handouts.
- Design your own animal and place at most 7 vertices on the animal. Then place edges between the vertices such that your result is a "tree":
  - By travel ling along edges, you can get from any vertex to any other vertex.
  - There are no loops.

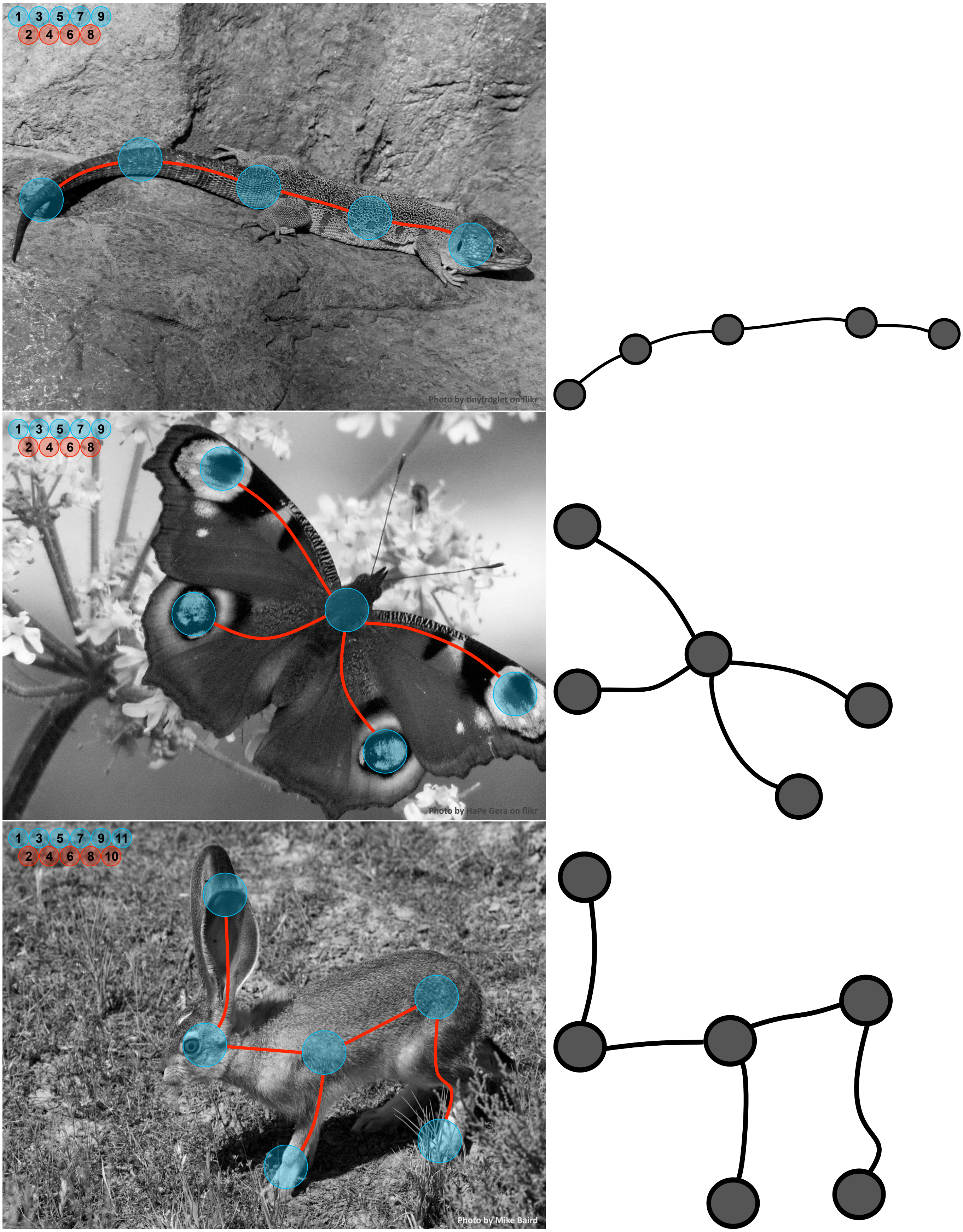
3.2 Handouts

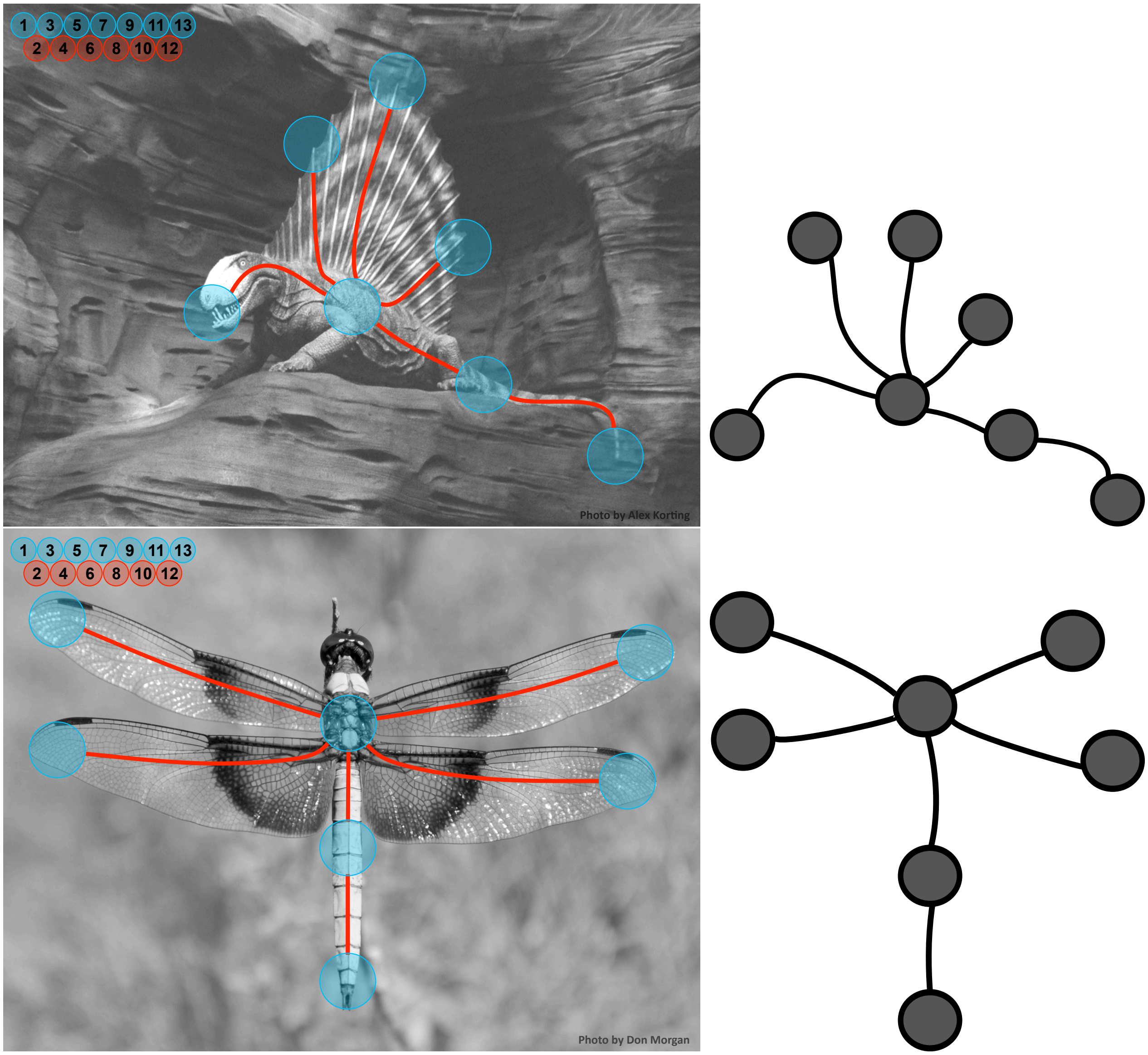









1 3 5 7 9
2 4 6 8
Photo by tinyfroglet on flikr
1 3 5 7 9
2 4 6 8
Photo by HaPe Gera on flikr
1 3 5 7 9 11
2 4 6 8 10
Photo by Mike Baird


1 3 5 7 9 11 13
2 4 6 8 10 12
Photo by Alex Korting
1 3 5 7 9 11 13
2 4 6 8 10 12
Photo by Don Morgan

3.3 The Mathematical Problem

The *Graceful Tree Conjecture* was proposed by Ringel, Kitzig and Rosa in 1967:

> Start with a tree and label the vertices with the consecutive odd numbers. Then, label the edges with the difference between the adjacent vertices. Can this always be done so that all edges are labeled differently?

3.4 More Details and More Ideas

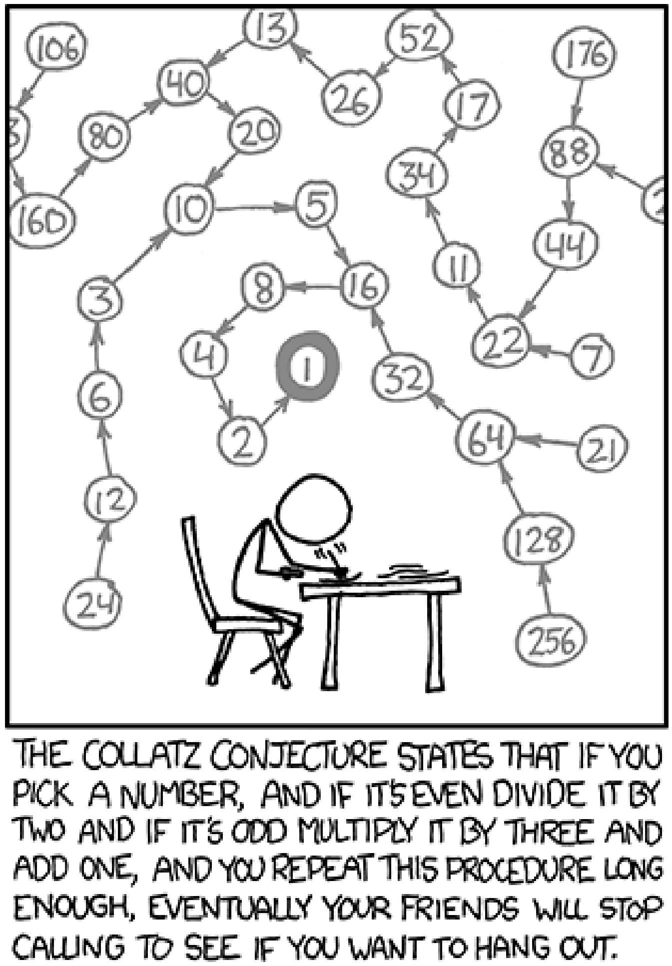
1. If you do some research, the graceful tree conjecture sounds a little different from what we have stated. Instead, you label the m vertices with the numbers $0, 1, 2, \ldots, m-1$ and label the edges with these differences and you want the edge labelings to be unique. Why is this the same as the definition we used?

2. The number of possible trees with a given number of vertices can be quite high. For example, there are nearly 110 billion trees with 32 vertices. Thus, if you wanted to test all these trees to see if they are graceful, then you would have to test an enormous number of trees. This has, in fact, been done for trees with 35 or less vertices: all trees with 35 or less vertices are graceful.

3. You could clearly investigate this with more and more vertices. But, it might also be interesting to allow non-connected trees (so you can't get from one vertex to every other vertex). Or, you could try it with graphs (so that there are loops).

4 Grade 4: Collatz Conjecture

Here is the basic problem, from [4]:



Gordan Hamilton does a nice job of integrating this problem into a story from Greek mythology.

4.1 The Activity

King Minos has just imprisoned Daedalus and his son Icarus in a high tower. Daedalus was implicated in the murder of the Minotaur, who happened to be King Mino's son.

Daedalus invents a way to escape. He and Icarus gather bee's wax and birds feathers to make wings to fly off the tower. The night before the flight, Icarus has a dream:

> He write a number on a rock, and he throws the rock off the tower. If the number on the rock is even, then it is halved. If the number on the rock is odd, then its tripled and one is added. Icarus continues this and if he ends up with 1 then he crashes into the sea.

Icarus knows that if he can do this and end up with something other than 1, then he will survive.

Daedalus has a similar dream, with one difference. For Daedalus, if the number is odd, then Daedalus triples the number and *subtracts* one. If Daedalus eventually ends up with 1, then he too crashes into the sea.

The task is to save Icarus and Daedalus.

4.2 The Mathematical Problem

The Collatz conjecture was first proposed by Lothar Collatz in 1937. It is unknown if the sequence for Icarus above must always end in a 1.

4.3 More Details and More Ideas

1. There are several cycles that will save Daedalus (find them!) but it is conjectured that Icarus cannot be saved.

2. The conjecture has been tested by a computer, for all starting values up to 5×2^{60}.

5 Grade 5: Play Perudo Perfectly

Perudo is a game and is sometimes called *Liar's Dice*.

5.1 The Activity

Each player starts with 5 dice. A player is randomly selected to be the first player.

1. Each player rolls his/her dice. Each player may see his/her own dice, but not the dice of the other players.

2. Starting with the first player, each player must either increase the bid or call "bluff" on the previous player's bid. A player is bidding on the total number of dice of a certain number. Here are some sample bids:

   - "25": I believe that there are at least 2 dice with 5's on them.
   - "35": I believe that there are at least 3 dice with 5's on them.

   In terms of bids we have: $25 < 35 < 36$.

3. If the player calls "bluff" correctly, then the player with that bid loses a dice.
   If the player calls "bluff" incorrectly, the calling player loses a dice.

4. The last player with dice left is the winner.

5.2 The Mathematical Problem

Perudo was played by Atahualpa (one of the last kings of the Incan empire) and Pizarro (who killed Atahualpa) in the 1530s:

> What is the best strategy to play Perudo?

5.3 More Details and More Ideas

1. Here is an already difficult simplification:

   > What if there are only two players, each with one dice each, what is the best strategy?

6 Grade 6: RSA Cipher

6.1 The Activity

This is a game with two players or teams. The players take turns selecting either prime or composite numbers as outlined on the board below. The key is that the product of the numbers chosen have to be equal. Here is a sample game:

After Move 1	Team A				Team B		
	Move 1	×	Move 3	=	Move 2	×	Move 4
	8	×		=		×	
	Composite Number		Prime Number		Prime Number		Composite Number

After Move 2	Team A				Team B		
	Move 1	×	Move 3	=	Move 2	×	Move 4
	8	×		=	2	×	
	Composite Number		Prime Number		Prime Number		Composite Number

After Move 3	Team A				Team B		
	Move 1	×	Move 3	=	Move 2	×	Move 4
	8	×	7	=	2	×	
	Composite Number		Prime Number		Prime Number		Composite Number

Team B can now win the game if he can find a composite number to make the equation true. If Team B can not make the equation equal, then Team A wins.

After Move 4	Team A				Team B		
	Move 1	×	Move 3	=	Move 2	×	Move 4
	8	×	7	=	2	×	28
	Composite Number		Prime Number		Prime Number		Composite Number

Keep the initial composite number 12 or less for the initial games. Then, move this to 50 or higher if you like.

Here are some questions:

- Which team is most likely to win? Is there a winning strategy?
- "Can I use a calculator?"

6.2 The Mathematical Problem

Break the RSA code, proposed by Rivest, Shamir and Aldeman in 1978:

> Find a reliable way for Team B to win.

6.3 More Details and More Ideas

Here is an example of how RSA really works (taken from [5]). We start with assuming Team B wants to send Team A a message (and that message is a number).

1. Team A selects two prime numbers. We will use $p = 23$ and $q = 41$ for this example, but keep in mind that the real numbers person A should use should be much much larger.

2. Team A multiplies p and q together to get $pq = (23)(41) = 943$. 943 is the "public key", which he tells to person B (and to the rest of the world, if he wishes).

3. Team A also chooses another number e which must be relatively prime to $(p1)(q1)$. In this case, $(p1)(q1) = (22)(40) = 880$, so $e = 7$ is fine. e is also part of the public key, so B also is told the value of e.

4. Now B knows enough to encode a message to A. Suppose, for this example, that the message is the number $M = 35$.

5. B calculates the value of $C = M^e \pmod{N} = 35^7 \pmod{943}$.

6. $35^7 = 64339296875$ and $64339296875 \pmod{943} = 545$. The number 545 is the encoding that B sends to A.

7. Now A wants to decode 545. To do so, he needs to find a number d such that $ed = 1 \pmod{(p1)(q1)}$, or in this case, such that $7d = 1 \pmod{880}$. A solution is $d = 503$, since $7 \cdot 503 = 3521 = 4 \cdot 880 + 1 = 1 \pmod{880}$.

8. To find the decoding, A must calculate $C^d \pmod{N} = 545^{503} \pmod{943}$. This looks like it will be a horrible calculation, and at first it seems like it is, but notice that $503 = 256+128+64+32+16+4+2+1$ (this is just the binary expansion of 503). So this means that

$$545^{503} = 545^{256+128+64+32+16+4+2+1} = 545^{256}545^{128}545^{1}$$

But since we only care about the result $\pmod{943}$, we can calculate all the partial results in that modulus, and by repeated squaring of 545, we can get all the exponents that are powers of 2. For example, $545^2 \pmod{943} = 545 \cdot 545 = 297025 \pmod{943} = 923$. Then square again: $545^4 \pmod{943} = (545^2)^2 \pmod{943} = 923^2 = 851929 \pmod{943} = 400$, and so on. We obtain the following table:

$$\begin{aligned}
5451 \pmod{943} =& 545\\
5452 \pmod{943} =& 923\\
5454 \pmod{943} =& 400\\
5458 \pmod{943} =& 633\\
54516 \pmod{943} =& 857\\
54532 \pmod{943} =& 795\\
54564 \pmod{943} =& 215\\
545128 \pmod{943} =& 18\\
545256 \pmod{943} =& 324
\end{aligned}$$

So the result we want is:

$$545^{503} \pmod{943} = 324 \cdot 18 \cdot 215 \cdot 795 \cdot 857\dot{4}00 \cdot 923 \cdot 545 \pmod{943} = 35$$

Using this tedious (but simple for a computer) calculation, A can decode B's message and obtain the original message $N = 35$.

- How does the above description of the RSA algorithm relate to the game we initially played?

7 Grade 8:

7.1 The Activity

Theseus goes into the labyrinth to slay the minotaur. He takes with him Ariadne's string, so that he can find his way out since he slays the minotaur. Theseus finally ends up in room with columns (see worksheets) and to slay the minotaur he must start at a particular column and wind the string from column to column. Each line segment he create (as defined by the string not making a corner) must be *longer* than the previous line segment. The more segments he is able to make, the better his chances are to slay the minotaur.

If Theseus does this in less than 8 steps, he is killed the minotaur. If he gets there in 8 steps, he kills the minotaur, but he is mortally wounded in the process. If he gets there in more than 8 steps, he kills the minotaur and gets the girl.

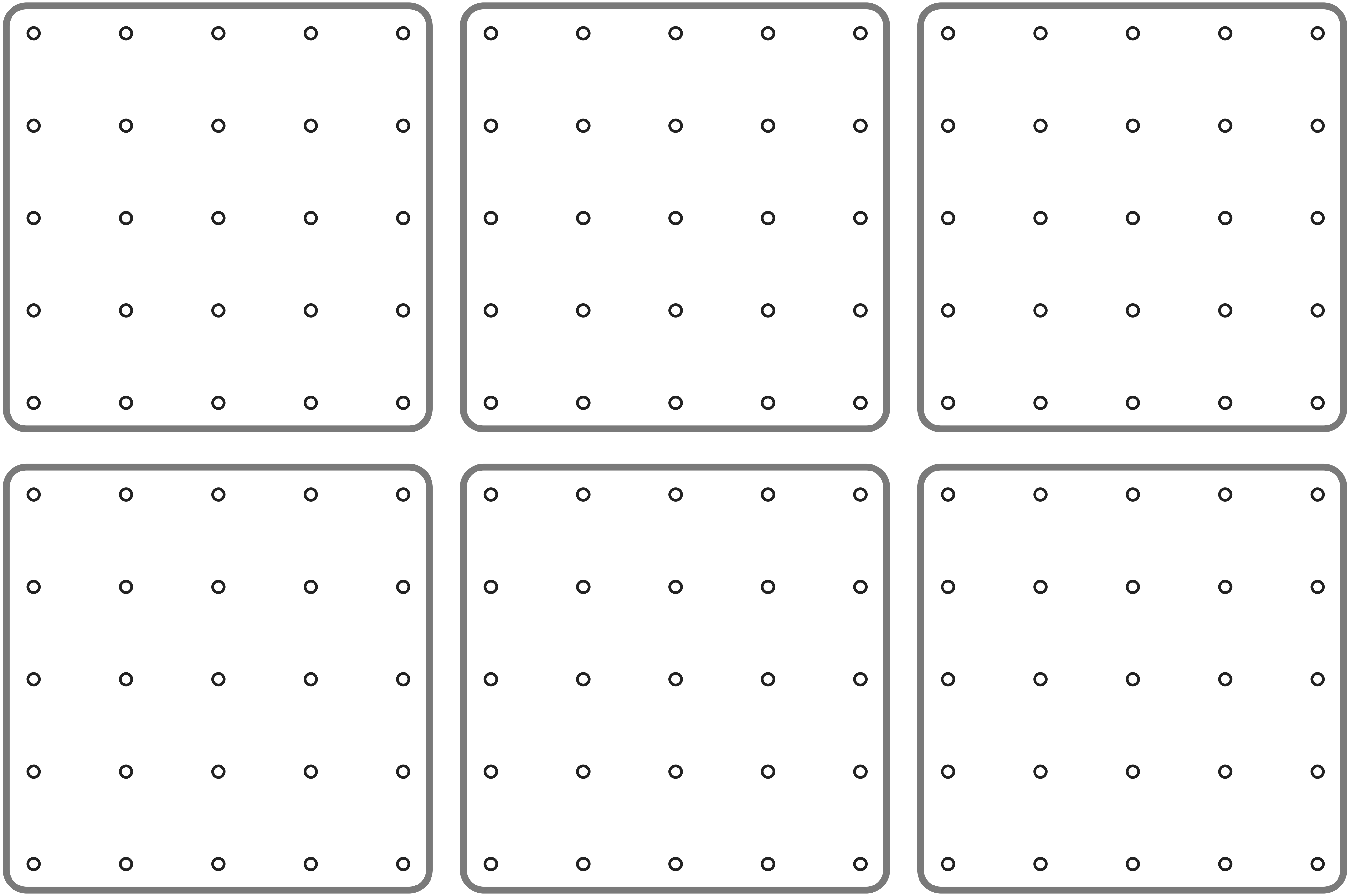
Start with the 5×5 problem and make it more complicated by moving to a 6×6, 7×7 or 8×8 grid.

7.2 The Mathematical Problem

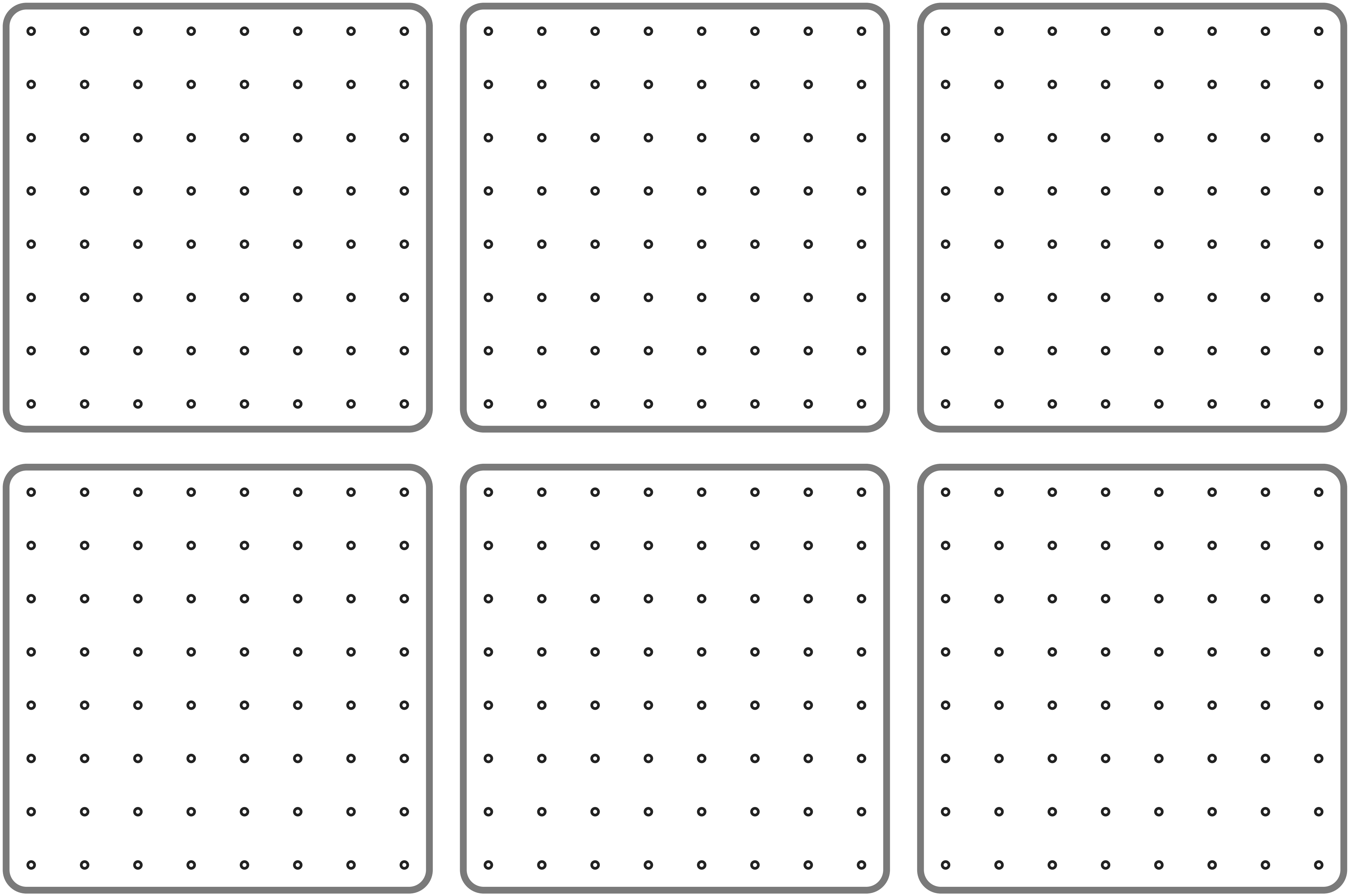
Given any n, make an $n \times n$ grid and try to find the maximum number of steps that can be taken.

7.3 Handouts

Ariadne’s String



Ariadne's String



8 Grades K, 7, 9-12:

There are currently no problems for grades K, 7 and 9-12.

References

[1] New Zealand Maths. *Strawberry Milk*, `http://www.nzmaths.co.nz/resource/strawberry-milk-0`.

[2] Wikipedia. *No-three-in-line problem*, `http://en.wikipedia.org/wiki/No-three-in-line_problem`.

[3] Hamilton, Gordon. *The MathPickle*, `http://www.mathpickle.com`.

[4] Munroe, Randall. *Collatz Conjecture*, `http://xkcd.com/710/`.

[5] Davis, Tom. *RSA Encryption*, `http://www.geometer.org/mathcircles/`.