1 Exercises: Place Value and Inequalities

1. Explain which numbers are greater.

   (a) $12,345$ or $432,243$

   (b) $12,345$ or $54,321$

   (c) $12,345$ or $14,321$

2. Imagine you have to explain to a fourth grader that $43 \times 100 = 4300$. How would you do it?

3. Imagine you have to explain to a fifth grader why $48 \times 500,000,000 = 24,000,000,000$. How would you do it?

4. What number should be added to $946,722$ to get $986,722$? What number should be added to $68,214,953$ to get $88,214,953$?

5. What number should be added to 58×10^4 to get 63×10^4 ? What number should be taken from 5210^5 to get 4810^5 ?

6. Which is bigger:

   (a) 4873 or 12001?

   (b) 4×10^5 or 3×10^6 ?

   (c) 8×10^{32} or 2×10^{33} ?

   (d) 4289×10^7 or 10^{11}?

   (e) $765,019,833$ or $764,927,919$?

7. Write each of the numbers 6100925730, 2300000000, and 7420000659 in expanded form.

8. Show that for any nonzero whole number k, $10^k > m \times 10^{k-1}$ for any single-digit number m.

2 Exercises: Laws of Operations

1. What would the associative law for addition look like for 4 terms? Is this necessary?

2. Why do we write AB instead of $A \times B$? When and how is it appropriate to introduce students to this?

3. (Activity) Verify by direct counting that the commutative law for multiplication is valid for M and N between 1 and 5 inclusive (in symbols: $1 \leq M, N \leq 5$), by using rectangular arrays of dots to represent multiplication of whole numbers.

4. (Activity) Verify by direct counting that the associative law for multiplication is valid for L, M, N between 1 and 5 inclusive (in symbols: $1 \leq L, M, N \leq 5$), by using 3-dimensional rectangular arrays of dots to represent triple products of whole numbers.

3 Exercises: Number Line and Laws

1. Draw a pictorial representation of $a < b$. Show where we see $a + c = b$ in the representation.

2. Convince yourself that $2a < 2b$ is the same as $a < b$, and that $3a < 3b$ is the same as $a < b$. Do it both numerically as well as on the number line.

3. Facts about inequalities. Explain why these are true.

   (a) $a + b < a + c$ is the same as $b < c$.

   (b) $a \neq 0$, $ab < ac$ is the same as $b < c$.

   (c) $a < b$ and $c < d$ implies that $a + c < b + d$.

   (d) $a < b$ and $c < d$ implies $ac < bd$.

4. Elaine has 11 jars in each of which she put 16 ping-pong balls. One day she decided to redistribute all her ping-pong balls equally among 16 jars instead. How many balls are in each jar? Explain.

5. Before you get too comfortable with the idea that everything in this world has to be commutative, consider the following.

   (a) Let $A_1 1$ stand for "put socks on" and A_2 for "put shoes on", and let $A_1 \circ A_2$ be "do A_1 first, and then A_2", and similarly let $A_2 \circ A_1$ be "do A_2 first, and then A_1". Convince yourself that $A_1 \circ A_2$ does not have the same effect as $A_2 \circ A_1$.

   (b) For any whole number k, let $B_1(k)$ be the number obtained by adding 2 to k, and $B_2(k)$ be the number obtained by multiplying k by 5. Show that no matter what the number n may be, $B_1(B_2(n)) \neq B_2(B_1(n))$.

6. Find shortcuts to do each of the following computations and give reasons (associative law of addition? commutative law of multiplication? etc.) for each step:

   (a) $833 + (5167 + 8499)$

   (b) $(54 + 69978) + 46$

   (c) $(25 \times 7687) \times 80$

(d) $(58679 \times 762) + (58679 \times 238)$

(e) $(4 \times 4 \times 4 \times 4) \times (5 \times 5 \times 5 \times 5 \times 5)$

(f) 64×125

(g) $(69 \times 127) + (873 \times 69)$

(h) $(([125 \times 24] \times 674) + ([24 \times 125] \times 326)$

7. Prove that both assertions in Problem 3 remain true if the strict inequality symbol "$<$" is replaced by the weak inequality symbol "$\leq$".

8. Let m and n be a 3 digit number and a 2 digit number, respectively. Can mn be a 4 digit number? 5 digit number? 6 digit number? 7 digit number?

9. Let m and n be a k digit number and an l digit number, respectively, where k and l and are nonzero whole numbers. How many digits can the number mn be? List all the possibilities and explain.

10. Suppose you have a calculator which displays only 8 digits, but you have to calculate 856164298×65. Discuss an efficient method to make use of the calculator to help with the computation. Explain.

    Do the same for 376241048×872.

11. Let x and y be two whole numbers.

    (a) Explain why $(x + y)(x + y) = x(x + y) + y(x + y)$.

    (b) Explain why $(x + y)(x + y) = xx + xy + yx + yy$.

    (c) Explain why $(x + y)(x + y) = x^2 + 2xy + y^2$, where as usual x^2 means xx and y^2 means yy.

12. Let x and y be two whole numbers. Explain why $(xy)(x+y) = (xy)x+(xy)y$.

4 Exercises: Addition

1. Write out each of the numbers in *expanded form*, perform the addition. Then, do the same operation using the standard algorithm.

   (a) $4,543 + 548$

   (b) $543,123 + 12,231$

   (c) $99,874 + 5,329$

2. Identify key points that can trip up your students. Come up with some problems that do not have these issues and come up with some problems that do.

3. How about for adding up more than two numbers. For example, what is the standard algorithm for $165 + 27 + 83 + 829$? Do this using the standard algorithm and by writing out the numbers in expanded form.

4. Explain to a fourth grader why the addition algorithm for $57032 + 2845$ is correct, first by writing it out in expanded form and then by using money (hundred dollar bills, 10 dollar bills, etc.).

5. Explain to a fourth grader why the addition algorithm $826 + 4907$ is correct, with and without the use of money.

6. Do the addition $67579 + 84937$ both ways: from left to right, and from right to left, and compare.

7. Compute $123 + 69 + 528 + 4$ by the addition algorithm, and give an explanation of why the computation is correct.

8. Compute $7826 + 7826 + 7826$ by the addition algorithm, and give an explanation of why the computation is correct.

9. Compute $670309 + 95000874$ by the addition algorithm, and give an explanation of why the computation is correct.

5 Exercises: Subtraction

1. Give an interpretation of $658 - 257$ in terms of money.

2. Explain to a fourth grader why the subtraction algorithm for 563 - 241 is correct, with or without money.

3. Give an interpretation of $756 - 389$ in terms of money.

4. Explain to a fourth grader why the subtraction algorithm for 627 - 488 is correct, with and without the use of money.

5. (a) Use the subtraction algorithm to compute $2403 - 876$ and explain why it is correct.

   (b) Do the same with $76431 - 58914$.

6. Compute $800,400 - 770,982$ in two different ways, and explain why what you have done is correct.

7. Compute $26,004 - 8325$ two ways, once using the standard algorithm, and once using the preceding "negative number" algorithm.

8. Let a, b, c be whole numbers.

   (a) Prove that $a + b < c$ is the same as $a < c - b$.

   (b) Suppose $c < a$ and $c < b$. Prove that $a < b$ is the same as $a - c < b - c$.

9. Find shortcuts to compute the following

   (a) $8 \times 875 =?$

   (b) $9996 \times 25 =?$

   (c) $103 \times 97 =?$

   (d) $86 \times 94 =?$

6 Exercises: Multiplication

1. Explain to a 4th grader why the multiplication algorithm for 86×37 is correct.

2. (a) Which 2-digit number, when multiplied by 89, gives a 4-digit number that begins and ends with a 6?

   (b) List all the 3-digit numbers which have the following properties: the sum of the digits is 12, and when multiplied by 15 they give a 5-digit number which ends with a 5 (i.e., the ones digit is 5).

   (Clearly this problem can be done by guess and check. You are however asked to use reasoning to quickly solve it by narrowing down the choices.)

3. Use the multiplication algorithm to compute and explain why it is correct.

$$\begin{array}{rcccccc} & & & & & 1 & 8 \\ \times & 5 & 0 & 0 & 0 & 9 & 2 \\ \hline \end{array}$$

   Compare to the algorithm applied to:

$$\begin{array}{rcccccc} & 5 & 0 & 0 & 0 & 9 & 2 \\ \times & & & & & 1 & 8 \\ \hline \end{array}$$

4. Compute 4208×879 by the multiplication algorithm and explain why it is correct.

7 Exercises: Measurement and Partitive Division

Definition 1. *(Measurement interpretation of Division)*
If $q = a \div b$ (so that $r = 0$) then we have

$$a = qb = b + b + \cdots + b \quad (q \ \textit{times})$$

This means that $a \div b$ is the total number of groups when a objects are partitioned into equal groups of b objects.

Because multiplication is commutative we can also write

Definition 2. *(Partitive Definition of Division)*
If $q = a \div b$ (so that $r = 0$) then we have

$$a = bq = q + q + \cdots + q \quad (b \ \textit{times})$$

This means that $a \div b$ is the number of objects in each group when a objects are partitioned into b equal groups.

Which definition of division is being used in each of the following problems.

1. Suppose you have 18 pennies and you want to make 6 equal groups.

2. Suppose you have 18 pennies and you want to put them into equal groups with 6 pennies in each group.

3. Suppose a car travels from town A to town B at a constant speed.

   (a) If the distance between the towns is 264 miles and it takes 4 hours, what is the speed?

   (b) If the speed is constant 58 mph and the distance is 58 miles, what is the time?

4. Make up your own problems.

8 Exercises: Division

1. (No calculator) Is 24 the quotient of $687 \div 27$?
2. (No calculator) Is 13 the quotient of $944 \div 46$?
3. (No calculator) Is 6977 the remainder of $124968752 \div 6843$?
4. (4 function calculator allowed) By taking multiples of the divisor, find the quotient and remainder in each of the following cases.
   (a) $964 \div 31$
   (b) $517 \div 19$
   (c) $6854 \div 731$
   (d) $4972086 \div 873$
   (e) $4972086 \div 659437$
5. Let r be the remainder of $a \div b$. Suppose $a = mA$ and $b = mB$ for some whole numbers m, A, B. Let R be the remainder of $A \div B$. What is the relationship between R and r
6. How would you explain to your class how to solve:

   > A faucet fills a bucket with water in 30 seconds and the capacity of the bucket is 12 gallons. How long would it take the same faucet to fill a vat with capacity 66 gallons.

7. Solve and discuss the following problems
   (a) If you try to put 234 gallons of liquid into 9 vats, with an equal amount in each vat, how much liquid is in each vat?
   (b) If you try to pour 234 gallons of liquid into buckets each with a capacity of 9 gallons, what is the minimum number of such buckets you need to hold these 234 gallons.

8.1 Exercises: Long Division

Some examples first:

$$
\begin{array}{cccccc}
 & & 1 & 9 & 5 \\
\hline
3 &) & 5 & 8 & 6 \\
 & & 3 & & \\
\hline
 & & 2 & 8 & 6 \\
 & & 2 & 7 & \\
\hline
 & & & 1 & 6 \\
 & & & 1 & 5 \\
\hline
 & & & & 1
\end{array} \tag{1}
$$

$$
\begin{aligned}
5 =& 1 \times 3 + 2 \\
28 =& 9 \times 3 + 1 \\
16 =& 5 \times 3 + 1
\end{aligned} \tag{2}
$$

$$
\begin{aligned}
586 =& 5 \times 10^2 + 8 \times 10 + 6 \\
=& (3 \times 1 + 2) \times 10^2 + 8 \times 10 + 6 \\
=& 3 \times 10^2 + 2 \times 10^2 + 8 \times 10 + 6 \\
=& (10^2 \times 3) + (20 \times 10) + 8 + 6 \\
=& (10^2 \times 3) + 28 \times 10 + 6 \\
=& (10^2 \times 3) + (3 \times 9 + 1) \times 10 + 6 \\
=& (10^2 \times 3) + (9 \times 10) \times 3 + 16 \\
=& (10^2 \times 3) + (9 \times 10) \times 3 + 5 \times 3 + 1 \\
=& (10^2 + 90 + 5) \times 3 + 1
\end{aligned} \tag{3}
$$

1. For each of the following do:

   - Long division
   - Step by step division (like in Equation (2)).
   - Expanded form long division (line in Equations (3)).

   (a) $58671 \div 3$

(b) $11546 \div 19$

(c) $642 \div 4$

(d) $10192 \div 8$

(e) $21850 \div 43$

(f) $50009 \div 67$

2. Explain to a sixth grader why the long division algorithm for $642 \div 4$ is correct.

3. Use Equations (2) and your previous work to derive the fact that $58671 = 3 \times 19557$. (In other words, we know you can multiply 3×19557 to get 19557, but use a sequence of division-with-remainders to explain the long division algorithm.)

4. Make up some more division problems and do them using all methods.

9 Activity: Other bases

Pick a positive whole number (5 is nice). Do everything now in base 5 (or your base that you picked. Go back and do all the other exercises in your new base. Talk about how the algorithms work, etc.