

# Clock Arithmetic

We are going to learn all about clock addition and the relationship to remainders when you divide numbers.

## 1 Standard Clock Addition

1. If it is 9 o'clock and you get out of school in 4 hours, when do you get out of school?
2. If it is 11 o'clock and you have to finish your math homework in 18 hours, what hour will it be at that time?
3. If it is 10 o'clock and you have to finish your math homework in 120 hours, what hour will it be at that time?
4. If it is 10 o'clock and you have to finish your math homework in 129 hours, what hour will it be at that time?
5. If it is 10 o'clock and you have to finish your math homework in 249 hours, what hour will it be at that time?
6. How many hours do you need to go past 5 o'clock to reach 12 o'clock.
7. How many hours do you need to go past 3 o'clock to reach 1 o'clock?
8. How many hours do you need to go past 6 o'clock to reach 2 o'clock?
9. How many hours before 4 o'clock was it 8 o'clock?
10. If the time reads 3:00 now, what will it read in 12 hours?

## 2 Clock addition and remainders

But what does all this have to do with remainders? Lets see if we can figure it out.

11. If it is 12:00 now, what time is it in 12 hours?
12. What is the remainder when you divide 12 by 12?
13. If it is 12:00 now, what time is it in 18 hours?
14. What is the remainder when you divide 18 by 12?
15. If it is 12:00 now, what time is it in 97 hours?
16. What is the remainder when you divide 97 by 12?
17. If it is 12:00 now, what time is it in 129 hours?
18. What is the remainder when you divide 129 by 12?
19. If it is 12:00 now, what time is it in 115 hours?
20. What is the remainder when you divide 115 by 12?
21. If it is 12:00 now, what time is it in 1200 hours?
22. What is the remainder when you divide 1200 by 12?
23. If it is 12:00 now, what time is it in 1205 hours?
24. What is the remainder when you divide 1205 by 12?

### 3 Exotic Clocks

Here we are going to work with clocks with the number of hours different from 12.

25. Suppose that we have a clock with 24 points, if the current time is 10:00, how long do we have to wait until it is 1:00 again?
26. Suppose that we have a clock with 24 points, if it was 19 o'clock twelve hours ago, what is the current time?
27. Suppose that we have a clock with 31 points, what time is 17 hours before 14?
28. Suppose that you have a clock with 9 points, if the current time is 3:00 and you look at your watch every 3 hours, what are the times you will see?
29. Suppose that you have a clock with 9 points, if the current time is 2:00 and you look at your watch every 2 hours, what times will you see?
30. Suppose that you have a clock with 9 points, if the current time is 4:00 and you look at your watch every 4 hours, what times will you see?
31. Suppose that you have a clock with 9 points, if the current time is 6:00 and you look at your watch every 6 hours, what times will you see?
32. Suppose that we have a clock with 45 points and that the clock currently reads 32. What will the clock read in 45 hours? What will the clock read in 0 hours?
33. Suppose that we have a clock with 4 points and that the clock currently reads 4:00. What time is it in 101 hours? What is the remainder when we divide 101 by 4?
34. Suppose that we have a clock with 4 points and that the clock currently reads 1:00. What time is it in 101 hours? What does this have to do with the previous question?
35. Suppose that we have a clock with 4 points and that the clock currently reads 2:00. What time is it in 101 hours? What does this have to do with the previous question?
36. Suppose that we have a clock with 4 points and that the clock currently reads 3:00. What time is it in 101 hours? What does this have to do with the previous question?

## 4 Addition Tables and Subtraction

Here is the addition table for a clock with 3 points

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

37. Write out the addition table for clocks with 4, 5, 6, and 12 points.
38. On a regular clock (with 12 points), subtracting 5 is the same as adding what number?
39. On a regular clock (with 12 points), what number should represent  $-5$  (negative 5)?
40. On a regular clock (with 12 points), subtracting 1 is the same as adding what number?
41. On a regular clock (with 12 points), what number should represent  $-1$  (negative 1)?
42. On a regular clock (with 12 points), subtracting 11 is the same as adding what number?
43. On a regular clock (with 12 points), what number should represent  $-11$  (negative 1)?
44. On a clock with 4 points, what number should represent  $-1$ ?
45. On a clock with 4 points, what number should represent  $-2$ ?
46. On a clock with 4 points, what number should represent  $-3$ ?
47. On a clock with 7 points, what number should represent  $-1$ ?
48. On a clock with 7 points, what number should represent  $-2$ ?
49. On a clock with 7 points, what number should represent  $-3$ ?
50. On a clock with 7 points, what number should represent  $-4$ ?
51. On a clock with 7 points, what number should represent  $-5$ ?
52. On a clock with 7 points, what number should represent  $-6$ ?

## 5 Multiplication and Division

Here is the multiplication table for a clock with 3 points

$\times$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

53. Write out the multiplication table for clocks with 4, 5, 6, 7, and 12 points.
54. If  $x$  is a number (a point on our clock) then the multiplicative inverse of  $x$  is the number  $y$  so that  $x \cdot y = 1$ .  
For clocks with 3,4,5,6,7 and 12 points find the multiplicative inverse of every number.
55. On a regular clock (with 12 points), dividing by 5 is the same as multiplying by what number?
56. On a regular clock (with 12 points), what number should represent  $\frac{1}{5}$ ?
57. In clock arithmetic does every number have a multiplicative inverse?  
Find a rule to determine which numbers do and do not have a multiplicative inverse.  
Find a rule for which clocks have more numbers that have multiplicative inverses.

## 6 Modular Arithmetic

This section is really just a note to tie everything together. Modular arithmetic is really just a fancy way a saying clock arithmetic. Instead of saying a clock with 7 points, mathematically one would write  $\mathbb{Z}_7$  and say “ $\mathbb{Z} \bmod 7$ ” or “the Integers mod 7.” Mathematically we would write things like

$$9 \equiv 2 \pmod{7}$$

As with all mathematics, theorems would be proven with this number system. Here are some examples of some theorems that might be helpful if you were going to study modular arithmetic in more depth. (Some of these theorems are “easy”, but some are difficult and deep. In any case, all of the theorems I listed below can be explored by examples.)

**Theorem 1.** *Suppose*

$$\begin{aligned} a_1 &\equiv a_2 \pmod{n} \\ b_1 &\equiv b_2 \pmod{n} \end{aligned}$$

*then the following are true*

$$\begin{aligned} a_1 + b_1 &\equiv a_2 + b_2 \pmod{n} \\ a_1 - b_1 &\equiv a_2 - b_2 \pmod{n} \\ a_1 b_1 &\equiv a_2 b_2 \pmod{n} \end{aligned}$$

**Theorem 2.** *Suppose  $ka \equiv kb \pmod{n}$  and  $\gcd(k, n) = d$ . Then  $a \equiv b \pmod{n/d}$ .*

**Theorem 3** (Fermat’s Little Theorem). *If  $p$  is a prime number then for any  $a$ :*

$$a^p \equiv a \pmod{p}$$

And, of course, it is always nice if mathematics is actually “useful.” If you write a base 10 number in “expanded form” and reduce  $\pmod{3}$  you get the divisibility by 3 test. (I’ll outline an argument for why this works.)

**Theorem 4.** *A number  $n$  is divisible by 3 if and only if the sum of its digits is divisible by 3.*

*Proof.* Oky, not really a proof, but more of an outline. Here are some steps that you would need to prove to understand why this works:

- i. A number  $n$  is divisible by 3 if and only if

$$n \equiv 0 \pmod{3}$$

- ii. For any  $k$ ,  $10^k \equiv 1 \pmod{3}$ .

- iii. Write out the number  $n$  in “expanded form” and reduce everything  $(\text{mod } 3)$  (well, we actually on reduce the powers of 10). And, to make this easier to see, suppose  $n$  is a 3-digit number with digits  $a, b, c$ , so we have  $n = \overline{abc}$  (I hope you understand what I mean by this.) Then,

$$\begin{aligned}n &= a \cdot 10^2 + b \cdot 10^1 + c \cdot 10^0 \\ &\equiv a \cdot 1 + b \cdot 1 + c \cdot 1 \pmod{3} \\ &\equiv a + b + c \pmod{3}\end{aligned}$$

- iv. Therefore  $n \equiv 0 \pmod{3}$  if and only if  $a + b + c \equiv 0 \pmod{3}$ .

□

Here are some problems that might be fun:

58. Find all solutions to the equation  $2x + 3 = 4$  in  $\mathbb{Z}_7$ .

59. Find all solutions to the equation  $2x + 3 = 4$  in  $\mathbb{Z}_6$ .

60. Write out addition and multiplication tables for  $\mathbb{Z}_{11}$ .

61. Find all solution to the following equations in  $\mathbb{Z}_{11}$ :

(a)  $3x - 4 = 9$

(b)  $2x^2 - 3x + 4 = 0$

(c)  $3x^2 - 4x + 2 = 0$

(d)  $4x^2 + 2x - 3 = 0$

62. Fix a prime number  $p$ . The *order* of a number  $x \pmod{p}$  is the smallest positive integer  $n$  such that  $x^n \equiv 1 \pmod{p}$ . So, for example, if  $p = 5$  then the order of  $2 \pmod{5}$  is 4 because  $2^4 \equiv 1 \pmod{5}$ .

Explore this idea by computing the order for all numbers  $\pmod{p}$ . Do this for different prime numbers. (I would suggest doing it for all prime numbers less than 20.)

What if you do this for a number that is not a prime number?

(This exploration should eventually lead to Fermat’s Little Theorem.)