Exam Rules:

1. Due in class on Friday Oct 2.
2. Work alone, use minimal sources (as discussed in class). Cite any sources you use.
3. You will be graded on your mathematics *and* your presentation of the mathematics. It must be readable or else I won't/can't read it!
4. Justify your answers.

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1. Consider the set $\mathbb{Z}[i] = \{a + ib | a, b \in \mathbb{Z}\}$. These are called the Gaussian Integers.
   Show that $\mathbb{Z}[i]$ is a commutative ring. To do this you must:
   (a) Show that $\mathbb{Z}[i]$ is closed under addition (given $z_1, z_2 \in \mathbb{Z}[i]$ then $z_1 + z_2 \in \mathbb{Z}[i]$).
   (b) Show that $\mathbb{Z}[i]$ is closed under multiplication (given $z_1, z_2 \in \mathbb{Z}[i]$ then $z_1 z_2 \in \mathbb{Z}[i]$).
   (c) Addition in $\mathbb{Z}[i]$ is commutative.
   (d) Addition in $\mathbb{Z}[i]$ is associtive.
   (e) Show that there is an identity for addition.
   (f) Multiplication in $\mathbb{Z}[i]$ is commutative (this is the commutative part of the *commutative* ring).
   (g) Multiplication in $\mathbb{Z}[i]$ is associtive.
   (h) Show that multiplication is distributive with respect to addition in $\mathbb{Z}[i]$.
2. Show that there is an identity for multiplication in $\mathbb{Z}[i]$.
3. A unit in $\mathbb{Z}[i]$ is an element that has a multiplicative inverse. Find all units in $\mathbb{Z}[i]$ (and prove you found all units).
4. In $\mathbb{Z}[i]$, definite the norm, $N(a + ib) = |a + ib|^2 = a^2 + b^2$. Show that if $z_1, z_2 \in \mathbb{Z}[i]$ then $N(z_1 z_2) = N(z_1)N(z_2)$.
5. Prove that $N(z) = 1$ if an only if z is a unit.
6. We will call a Gaussian integer to be Gaussian prime if it not the product of Gaussian integers of smaller norm. Determine (with proof) if the following Gaussian integers are Gaussian prime or not.
   (a) $4 + i$
   (b) 2
   (c) $1 - i$
   (d) 3
   (e) 17
7. Find all Gaussian primes with norm less than 50.
8. Prove that any Gaussian integer factors into Gaussian primes.
   (You don't have to prove anything about uniqueness.)
9. Let $p \in \mathbb{N}$ be a prime (a prime in the usual sense). Prove the p is also prime in $\mathbb{Z}[i]$ if and only if p is not the sum of two squares.
   (Hint?: In my solution I prove the equivalent statement: p is a sum of two squares if and only if p is not prime in $\mathbb{Z}[i]$.)
10. State and prove a division algorithm for $\mathbb{Z}[i]$.
    (It should be very similar to the division algorithm for $\mathbb{Z}$, don't worry about the uniqueness part of this.)
11. Find all 12th roots of unity (written out in $a + bi$ format).
12. Find all *primitive* 12th roots of unity.