Name:

- Show all your work.
- You will be graded on your ability to make your work and ideas clear.
- You may use any result used in class or from the textbook.
- No note cards, calculators allowed.
- There are 6 problems on TWO pages. Each problem is worth 10 points.

1. State the Cauchy-Schwartz inequality.
   
   **Solution:** For \( x, y \in \mathbb{R}^n \),
   
   \[ |x \cdot y| \leq \|x\| \|y\| \]
   
   And, equality holds if and only if one of the vectors is a scalar multiple of the other.

2. Let \( v \in \mathbb{R}^2 \) be \( v = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \) and
   
   \[ f \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{cases} \frac{x^2y}{x^2 + 2y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases} \]
   
   Find \( D_v f \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) \).
   
   **Solution:**
   
   \[ D_v f(0) = \lim_{t \to 0} \frac{f(0 + tv) - f(0)}{t} = \lim_{t \to 0} \frac{1}{t} \cdot \frac{4t^3}{t^2(4 + 2)} = \frac{4}{6} = \frac{2}{3} \]

3. Find the limit and prove you are correct (using \( \varepsilon \)).
   
   \[ \lim_{n \to \infty} \frac{n}{4n + 7} \]
   
   **Solution:**
   
   \[ \lim_{n \to \infty} \frac{n}{4n + 7} = \frac{1}{4} \]
   
   **Proof.** Let \( \varepsilon > 0 \). Let \( N \in \mathbb{N} \) be such that \( N > \frac{7}{16\varepsilon} \). Then, if \( n > N \) we have
   
   \[ 4N + 7 > 4 \cdot \frac{7}{16\varepsilon} + 7 > \frac{7}{4\varepsilon} \]
   
   Rearranging gives
   
   \[ \frac{7}{4(4N + 7)} < \varepsilon \]
Therefore,
\[
\left| a_n - \frac{1}{4} \right| = \left| \frac{n}{4n+7} - \frac{1}{4} \right| = \left| \frac{4n - (4n + 7)}{4(4n + 7)} \right| = \frac{7}{4(4n + 7)} < \frac{7}{4(4N + 7)} < \varepsilon
\]

4. Given an example of a function, \( f : \mathbb{R}^n \to \mathbb{R}^m \) that is continuous but not differentiable.

   (a) Using \( \varepsilon \) and \( \delta \), prove that your function is continuous.

   (b) Prove, using our definition of differentiable, that your function is not differentiable. You are not required to use \( \varepsilon \) or \( \delta \)'s for this.

   (Hint: You are free to choose \( n \) and \( m \), perhaps \( n = m = 1 \) is easiest?)

   **Solution:** There are many examples but perhaps the easiest function is

   \[ f(x) = |x| \]

   First, this is continuous at all points. To see that the function is continuous at \( a \in \mathbb{R} \), let \( \varepsilon > 0 \) be given. Then, let

   \[ \delta = \varepsilon \]

   Then, if \( |x - a| < \delta \) then, by the triangle inequality,

   \[ ||x| - |a|| \leq |x - a| < \varepsilon \]

   Thus, \( f \) is continuous.

   Now, to see that \( f \) is not differentiable at 0, suppose \( Df(0) = c \) for some \( c \in \mathbb{R} \). Then,

   \[
   \lim_{h \to 0} \frac{f(0+h) - f(0) - ch}{|h|} = \lim_{h \to 0} \frac{|h| - ch}{|h|} = \lim_{h \to 0} 1 - c \frac{h}{|h|}
   \]

   We can look at the left and right hand limits to see that this can not exist

   \[
   \lim_{h \to 0^+} 1 - c \frac{h}{|h|} = \lim_{h \to 0^+} 1 - c \frac{h}{h} = 1 - c
   \]

   \[
   \lim_{h \to 0^-} 1 - c \frac{h}{|h|} = \lim_{h \to 0^-} 1 + c \frac{h}{h} = 1 + c
   \]

   Thus, no matter what value for \( c \) is chosen, this limit can never be equal to 0 and \( f \) is not differentiable at 0.

5. Let \( g : \mathbb{R}^3 \to \mathbb{R} \) and \( f : \mathbb{R} \to \mathbb{R}^2 \) be defined as

   \[
   g \begin{pmatrix} x \\ y \\ z \end{pmatrix} = xy + z^2y - 5
   \]

   \[
   f(t) = \begin{pmatrix} t^2 \\ 1 - t \end{pmatrix}
   \]
(a) Using the chain rule, find

\[
D(f \circ g) \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}
\]

Solution: We compute:

\[
g \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = -2
\]

\[
f (-2) = \begin{bmatrix} 4 \\ 3 \end{bmatrix}
\]

\[
Dg \begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \ x + z^2 \ 2yz
\]

\[
Dg \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}
\]

\[
Df (t) = \begin{bmatrix} 2t \\ -1 \end{bmatrix}
\]

\[
Df (-2) = \begin{bmatrix} -4 \\ -1 \end{bmatrix}
\]

Therefore

\[
D(f \circ g) \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = Df(-2) \circ Dg \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}
\]

\[
= \begin{bmatrix} -4 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ -12 \\ -16 \end{bmatrix}
\]

(b) Using the derivative, approximate

\[
(f \circ g) \begin{pmatrix} -1.5 \\ 0.75 \\ 2.5 \end{pmatrix}
\]

Solution: Letting \( a = [-1, 1, 2]^T \) and \( h = [-0.5, -0.25, 0.5]^T \),

\[
f(a + h) \approx f(a) + D(f \circ g)h = \begin{bmatrix} 4 \\ 3 \end{bmatrix} + \begin{bmatrix} -4 \\ -1 \end{bmatrix} + \begin{bmatrix} -12 \\ -3 \end{bmatrix} + \begin{bmatrix} -16 \\ -4 \end{bmatrix} = \begin{bmatrix} -0.5 \\ -0.25 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix} + \begin{bmatrix} -3 \\ -0.75 \end{bmatrix} = \begin{bmatrix} 1 \\ 2.25 \end{bmatrix}
\]

6. True or False. Justify your answer (with either a proof, reference to a theorem, or counterexample).
(a) The set 
\[ S_1 = \left\{ \frac{1}{2n+1} \mid n \in \mathbb{N} \right\} \subset \mathbb{R} \]
is closed.
Solution: False. This set is not closed nor open. There are several ways to see this but perhaps the easiest is to consider the sequence \( \left\{ \frac{1}{2n+1} \right\}_{n=1}^{\infty} \subset S_1 \). This is a convergent sequence, converging to 0, but the convergent point is not in \( S_1 \), violating the definition of closed (or, violating the theorem we did in class about this).

(b) The set 
\[ S_2 = \{(x, y) \in \mathbb{R}^2 \mid y = 0\} \]
is closed.
Solution: This is true. To see this, take \((x, y) \notin S_2\). Then, \( y \neq 0 \) and \( B((x, y), |y|) \) is completely contained in the complement of \( S_2 \).

(c) Let \( f : \mathbb{R}^n \to \mathbb{R}^m \) be continuous. Then for every open set \( U \subset \mathbb{R}^n \), the set \( f(U) \), defined as usual by
\[ f(U) = \{y \in \mathbb{R}^m \mid y = f(x) \text{ for some } x \in U\} \subset \mathbb{R}^m \]
is also open.
Solution: This is false. A function that has this property is called an open map. Here is one function that is not open, but there are many:
\[ f(x) = x^2, \quad U = (-1, 1) \]

(d) Let \( f : \mathbb{R}^n \to \mathbb{R}^m \). Suppose that all first partial derivatives of \( f \) exist at all points in the domain, so that the Jacobian of \( f \), \( Jf \), exist all all points. Then the derivative of \( f \) is given by
\[ Df(a) = Jf(a) \]
Solution: This is false, but it would be true if we also required first partials to be continuous. So, the example should be something like we did in class:
\[ f(x, y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases} \]