Problem Solving Methods

Blake Thornton

One of the main points of problem solving is to learn techniques by just doing problems. So, lets start with a few problems and learn a few techniques.

Patience

1. Find a 9-digit number using each digit 1 through 9 once, such that the first n digits are divisible by n

   Solution: You can ask the corresponding question for numbers with fewer digits:

   > Find an m digit number ($m \leq 9$) using each digit 1 through m once, such that the first n digits are divisible by n.

   Here's what you find:

Digits, m	**Number**
1	1
2	12
3	$123, 321$
4	None
5	None
6	$123654, 321654$
7	None
8	38165472
9	381654729

2. Sheep and Wolves. On a 5×5 chessboard place 5 wolves (who move like chess queens) and 3 sheep so that the sheep are safe from being eaten by the wolves.

   Solution:

S			S	
S				
				W
	W			W
	W	W		

   You might ask about other size boards or different numbers of sheep.

Size	(wolves, sheep)
3×3	(1,2), (2,1)
4×4	(1,6), (2,3), (3,2),(4,1),(5,1),(6,1)
5×5	(1,12), (2,7), (3,5), (4,4)
6×6	(1,20), (2,13), (3, 9), (4, 8), (5,6)
8×8	(1,42), (2,31), (3,25), (4,24), (5,17), (6,15), (7,14?), (8,11?)

Here is an 8×8 board with 8 wolves:

W	W	W					
	W	W	W				
					S	S	S
						S	S
							S
W	W						
				S	S		
				S	S	S	

Can you find a configuration of 8 wolves on an 8×8 board with more sheep than this?

3. Triangle problem: arrange the numbers 1 through 6 into a "difference triangle" where each number in the row below is the difference of the two numbers above it. For example

$$\begin{array}{ccccc} 6 & & 4 & & 1 \\ & 2 & & 3 & \\ & & 1 & & \end{array}$$

almost works but it has two 1's and no 5.

How about with 10 numbers? 15?

Solution: There are 8 different possibilities for the 1 through 6 triangle. Here are a couple

$$\begin{array}{ccccc} 1 & & 6 & & 4 \\ & 5 & & 2 & \\ & & 1 & & \end{array} \qquad \begin{array}{ccccc} 2 & & 6 & & 5 \\ & 4 & & 1 & \\ & & 2 & & \end{array} \qquad \begin{array}{ccccc} 6 & & 1 & & 4 \\ & 5 & & 3 & \\ & & 2 & & \end{array}$$

For the 1 through 15 triangle there are also 8 possibilities a couple of which are

$$\begin{array}{ccccccc} 6 & & 1 & & 10 & & 8 \\ & 5 & & 9 & & 2 & \\ & & 4 & & 7 & & \\ & & & 3 & & & \end{array} \qquad \begin{array}{ccccccc} 6 & & 1 & & 10 & & 8 \\ & 5 & & 9 & & 2 & \\ & & 4 & & 7 & & \\ & & & 3 & & & \end{array} \qquad \begin{array}{ccccccc} 8 & & 3 & & 10 & & 9 \\ & 5 & & 7 & & 1 & \\ & & 2 & & 6 & & \\ & & & 4 & & & \end{array}$$

Try special cases (make up an easier problem!)

4. How many zeros are at the end of the number

$$100! = 100 \cdot 99 \cdot 98 \cdot 97 \cdots 3 \cdot 2 \cdot 1$$

Solution: Here we can see that a zero at the end comes from having a factor of 2 and 5. Obviously there will be a surplus of factors of 2, so we just have to count the factors of 5. It turns out that 100! has 24 zeros at the end.

Note, you probably do not want to actually do this computation! Here it is though (158 digits long):

$$
\begin{aligned}
100! = \\
&93326215443944152681699238856266700490715968264381\\
&62146859296389521759999322991560894146397615651828\\
&6253697920827223758251185210916864\\
&000000000000000000000000
\end{aligned}
$$

5. The numbers 1 through 100 are written on the board. Take two numbers, u and v and erase them writing $uv + u + v$ in their place. This leaves 99 numbers. Keep doing this and after a while, there will only be one number left on the board. What are the possible numbers left?

   Solution:

   (a) Note that the formula for replacing is symmetric—it does not matter which number is first.

   (b) What happens if there are only three numbers on the board? Show that there there will only be one possible number left.

   $$a \quad b \quad c \quad \longrightarrow \quad abc + ab + ac + bc + a + b + c$$

   (c) Note the pattern above for three numbers. Here's what happens for four numbers:

   $$a \quad b \quad c \quad d \longrightarrow \quad abcd+abc+abd+acd+bcd+ab+ac+ad+bc+bd+cd+a+b+c+d$$

   (d) Conjecture that there will be a unique number left at the end. All the ingredients to prove the conjecture are above (induction required).

   If you actually wanted to compute the one number? You would most certainly want to use a computer. I get the following 160 digit number:

   $$
   \begin{aligned}
   &94259477598383594208516231244829367495623127947025437683278893\ldots\\
   &\ldots 53416977599316221476503087861591808346911623490003549 5\ldots\\
   &\ldots 995833697063026032639999999999999999999999999
   \end{aligned}
   $$

Getting dirty

6. What is the smallest number that can not be written by subtracting a prime from a square. For example

$$
\begin{aligned}
1 &= 4 - 3 \\
2 &= 9 - 7 \\
3 &= ?
\end{aligned}
$$

(How about the next smallest number?)

Solution: Here trial and error goes a long way, just try it. You will soon find:

$$
\begin{aligned}
1 &= 4 - 3 \\
2 &= 9 - 7 \\
3 &= 16 - 13 \\
4 &= 9 - 5 \\
5 &= 16 - 11 \\
6 &= 9 - 3 \\
7 &= 9 - 2 \\
8 &= 25 - 17 \\
9 &= 16 - 7 \\
10 &= 81 - 71 \\
11 &= 16 - 5 \\
12 &= 25 - 13 \\
13 &= 16 - 3 \\
14 &= 16 - 2 \\
15 &= 196 - 181 \\
16 &= \textbf{None} \\
17 &= 36 - 19 \\
18 &= 25 - 7 \\
19 &= 36 - 17 \\
20 &= 25 - 5
\end{aligned}
$$

It looks like we are unable to find numbers that work for 16 but this could be because we did not look hard enough.

Lets see if we can prove that there is no such number possible. Suppose $16 = m^2 - p$ where p is prime. Then we have $p = m^2 - 16 = (m-4)(m+4)$. This can only happen if $m - 4 = 1$ or $m = 5$ (since p is prime) which would mean that $p = 5 + 4 = 9$ which is not prime.

To find the next smallest number we might guess that it should probably be a perfect square (since this is why 16 didn't work). So, we can try perfect squares and see that the next numbers that do not work are: $49 = 7^2$ and $100 = 10^2$.

It is not too much work to show that if $n = a^2$ then there is no m and no p prime such that $n = m^2 - p$ if $2a + 1$ is not prime.

7. For every positive integer n, look at the number $n^3 - n$. The first few are here:

n	$n^3 - n$
1	0
2	6

   Keep filling out this chart. For at least the first few numbers in the $n^3 - n$ column, they should be divisible by 3.

   (a) Are all the numbers $n^3 - n$ divisible by 3?

   (b) If not, find one that is not. If so, show that this is always the case.

   Solution: Note that $n^3 - n = n(n-1)(n+1)$, the product of three consecutive integers. Therefore, at lease one of $n - 1$, n, $n + 1$ that is divisible by 3.

8. For every positive integer n, look at the number $n^5 - n$. The first few are here:

n	$n^5 - n$
1	0
2	40

   Keep filling out this chart. For at least the first few numbers in the $n^5 - n$ column, they should be divisible by 5.

   (a) Are all the numbers $n^5 - n$ divisible by 5?

   (b) If not, find one that is not. If so, show that this is always the case.

   Solution: Note that $n^5 - n = n(n-1)(n+1)(n^2+1)$, which is not the product of five consecutive integers. But notice that if n is a number that ends in $0, 1, 4, 5, 6, 9$ then one of $n - 1$, n, $n + 1$ is divisible by 5. If n ends in 2,3,7,8 then $n^2 + 1$ is divisible by 5.

9. How many rectangles are in a 10×10 rectangle?

   Solution: Can you count how many rectangles have a given spot as their lower right corner? Where can the top left corner be?