- You may use a calculator and you may have a 3×5 note card of notes.
- Justify your answers with your work. Answers alone may be worth 0 points.
- There are 14 questions, each worth 10 points each.

1. Demonstrate, using exploding dots, $23_5 + 443_5$.

   Solution: $23_5 + 443_5 = 1021_5$

2. Convert 123_{10} into base 7.

   Solution: $123_{10} = 234_7$

3. (a) Find the last digit of 3^{2013}.

   Find the pattern:

$$\begin{aligned} 3^1 &\equiv 3 \pmod{10} \\ 3^2 &\equiv 9 \pmod{10} \\ 3^3 &\equiv 7 \pmod{10} \\ 3^4 &\equiv 1 \pmod{10} \\ 3^5 &\equiv 3 \pmod{10} \\ 3^6 &\equiv 9 \pmod{10} \\ 3^7 &\equiv 7 \pmod{10} \\ 3^8 &\equiv 1 \pmod{10} \end{aligned}$$

   So, we notice that it repeats every 4, and $3^4 \equiv 1$. Also, $2013 = 4 \cdot 503 + 1$:

$$3^{2013} = 3^{4 \cdot 503+1} = 3^{4 \cdot 503} \cdot 3^1 = (3^4)^{503} \cdot 3 \equiv 1^{503} \cdot 3 \pmod{10}$$

   Thus, the last digit is a 3.

   (b) Find the last digit of 3^{2013} when the number is written in base 3.

   Here we went the remainder of the number when divided by 3, which is 0.

4. Find the remainder of 75424_7 (this is written in base 7) when divided by 7.

   Solution: If you think of this as exploding dots, notice that all the dots to the left of the ones digit will become a multiple of 7 dots. Thus, the remainder must be what is left in the ones digit, or 4.

5. Let $A = \{\text{cat}, \text{dog}, \text{mouse}\}$ and $B = \{\text{dog}, \text{zebra}\}$. Write out all the elements in the following sets:

   (a) $A \cap B = \{\text{dog}\}$

   (b) $A \cup B = \{\text{cat}, \text{dog}, \text{mouse}, \text{zebra}\}$

   (c) $A - B = \{\text{cat}, \text{mouse}\}$

   (d) $B - A = \{\text{zebra}\}$

   (e) $A \times B = \{(\text{cat}, \text{dog}), (\text{cat}, \text{zebra}), (\text{dog}, \text{dog}), (\text{dog}, \text{zebra}), (\text{mouse}, \text{dog}), (\text{mouse}, \text{zebra})\}$

   (f) $B \times A = \{(\text{dog}, \text{cat}), (\text{dog}, \text{dog}), (\text{dog}, \text{mouse}), (\text{zebra}, \text{cat}), (\text{zebra}, \text{dog}), (\text{zebra}, \text{mouse})\}$

6. Let $A = \{1, 2, 3\}$ and $B = \{x, y, z\}$. How many different one to one correspondences are there between A and B? Find them all.

   Solution: There are six different one to one correspondences:

1	x
2	y
3	z

1	x
2	z
3	y

1	y
2	x
3	z

1	y
2	z
3	x

1	z
2	x
3	y

1	z
2	y
3	x

7. List all solutions to:

$$\{1, 2\} \subseteq X \subseteq \{1, 2, 3, 4, 5\}$$

   Solution: Here are all the possible solutions, there are 8 solutions:

$$\{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{1, 2, 4, 5\}, \{1, 2, 3, 4, 5\}$$

8. Suppose A has 13 elements, B has 17 elements, and $A \cup B$ has 23 elements. How many elements are in $A \cap B$?

   Solution: Draw a Venn Diagram or use the formula $|A \cup B| = |A| + |B| - |A \cap B|$. In any case, $|A \cap B| = 7$.

9. Determine which, if any, of the following sets are closed under multiplication. Justify your answers.

   (a) $A = \mathbb{N} - \{4\}$

   Solution: Not closed under multiplication since $2 \times 2 = 4 \notin A$.

   (b) $B = \mathbb{N} - \{7, 13\}$

   Solution: Closed under multiplication. $\mathbb{N}$ is closed under multiplication. If we take a prime number out of $\mathbb{N}$, there is no way to multiply to get that missing prime number.

10. True or False. A and B below are sets. Justify your answer. (Recall that $\sim$ means that the sets can be put into one to one correspondence.)

    (a) If $A = B$ then $A \sim B$.

    Solution: True, a set can be put into one to one correspondence with itself.

    (b) If $A \sim B$ then $A = B$.

    Solution: False. Here is an example where this fails. Let $A = \{1\}$ and $B = \{2\}$. $A \sim B$ but $A \neq B$.

    (c) $(A \cap B) \subseteq (A \cup B)$.

    Solution: True. Every element in the intersection of A and B is also in the union of A and B.

    (d) Every prime number is odd

    Solution: False. Every prime number except 2 is odd.

    (e) Larger numbers have more factors than smaller numbers.

    Solution: False. Large numbers can still be prime (and thus have only 2 factors).

11. Perform the following divisions and write your answer in the format of the Division Algorithm.

    (a) $93 \div 4$

    Solution: $93 = 23 \times 4 + 1$

    (b) $4 \div 93$

    Solution: $4 = 0 \times 93 + 4$

12. (a) Rewrite the following with a single exponent:

    $$(3^8 \cdot 9^5 \cdot 81^4) \div 27^2 = 3^{28}$$

    (b) If $2^{x+1} = 10$ then what is 2^{2x+3}?

    Solution: Since $2^{x+1} = 2^x \cdot 2$, we have $2^x = 5$.

    $$2^{2x+3} = (2^x)^2 \cdot 2^3 = 5^2 \cdot 8 = 200$$

13. Use the Euclidean Algorithm to find $\gcd(1456, 8463)$.

    Solution: $\gcd(8463, 1456) = 91$

    $$\begin{aligned} 8463 &= 5 \cdot 1456 + 1183 \\ 1456 &= 1 \cdot 1183 + 273 \\ 1183 &= 4 \cdot 273 + 91 \\ 273 &= 3 \cdot 91 + 0 \end{aligned}$$

14. Determine if it is possible to find integers x and y to solve the equations below. If it is possible, you are not required to find x and y, but you will need to justify your answer in either case.

    Solution: The general result is that there exist integer solutions to $ax + by = c$ if and only if c is a multiple of $\gcd(a, b)$. Thus, we only have to look at the $\gcd(a, b)$.

    (a) $35x + 22y = 1$

    Solution: It is possible because $\gcd(35, 22) = 1$.

    (b) $35x + 21y = 1$

    Solution: It is not possible because $\gcd(35, 21) = 7$.