Exploding Dots

Blake Thornton

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Most of this is due directly to James Tanton (`http://www.jamestanton.com/`) and Linda Green (`http://www.marinmathcircle.org/`).

> **There are 10 types of people in the world. Those that understand binary and those that don't.**
>
> **There are 10 types of people in the world. Those who understand ternary, those who don't, and those who mistake it for binary**

1. What is the smallest number of integral weights needed so that you can weight any integer number of grams of chocolate from 1 to 100 on a balance scale. Weights are only allowed to be placed on one side and the chocolate on the other side.

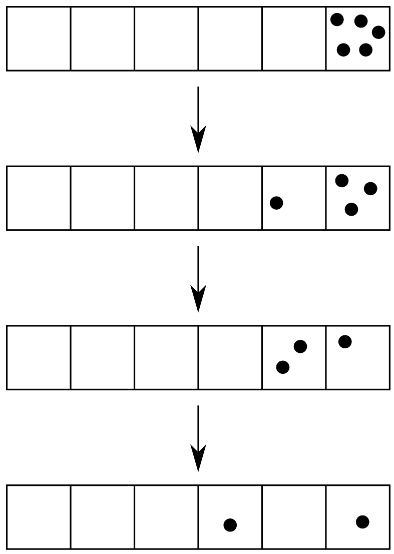
2. What if, in Problem 1, you were allowed to put weights in either side of the balance?

Exploding Dot Machines

A $1 \leftarrow 2$ machine consists of a row of boxes, extending to the left. To operate the machine, place a number of dots in the right most box. The machine then redistributes the dots according to the rule:

> Two dots in any one box vanish (they explode) and are replaced with one dot one box to their left.

When all the explosions have died down, you can read off a code of 1's and 0's representing the number of dots in each box.



3. Run the $1 \leftarrow 2$ machine several times. At a minimum, run the machine starting with the following amounts: 6, 8, 25, 32.
4. You have some choices in the order that you do the explosions. Does this order affect the final code?
5. Run a $1 \leftarrow 3$ machine. Find the codes for 13, 14, 15, 20.
6. Encode 19, 42, and 100 using a $1 \leftarrow 6$ machine.
7. Suppose the code 152 was obtained using a $1 \leftarrow 6$. How many dots were there to start with? What if the code was 13251?
8. What is the code for 254 using a $1 \leftarrow 10$ machine?

Number Bases

A number written with a subscript will mean that that the subscript number is the base. In other words, you would read 11_6 as 11 base 6 and this means that this is the coded number in a $1 \leftarrow 6$ machine (which, is the usual number 7). A number without a subscript means it is encoded in base 10. So, 27 and 27_{10} are the same thing.

9. Write these numbers in base 10:
   (a) 15_7
   (b) 35_7
   (c) 45_7
   (d) 412_7

10. Write these numbers in base 7:
    (a) 13
    (b) 48
    (c) 63
    (d) 625
    (e) 1000

11. Write these numbers in base 3:
    (a) 13_6
    (b) 24_5
    (c) 1011_2
    (d) 71_9

12. (a) How many different symbols (digits) are needed when writing numbers base 10?
    (b) How many different symbols (digits) are needed when writing numbers base 2?
    (c) How many different symbols (digits) are needed when writing numbers base n?

13. Write the number 100_{10} in base 2, base 3, base 4, base 5, base 6, base 7, base 8, base 9.

14. Try writing 2013_{10} in base 12.

15. Count to a hundred in base 12. Try counting to a thousand in base 3.

More on Bases

16. Explain the following statement:

    There are 10 types of people in the world. Those that understand binary and those that don't.

17. Explain:

    There are 10 types of people in the world. Those who understand ternary, those who don't, and those who mistake it for binary

18. Some initial practice. Fill in the chart so that the representations in each row are of the same number.

Base 2	**Base 5**	**Base 10**	**Base 12**	**Base 16**
1011_2				
		41_{10}		
			$1AB8_{12}$	
				$F01BC_{16}$
10001_2				
	4431_5			
				1011
				C

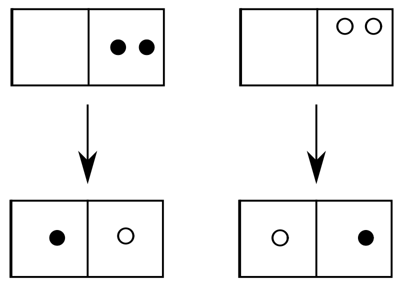
Crazy Dot Machines

19. Experiment with a $2 \leftarrow 3$ machine. Be sure to encode the numbers 1 through 30. Determine what base is represented by a $2 \leftarrow 3$ machine.

20. Experiment with a $2 \leftarrow 4$ machine. Work on conversion between the $1 \leftarrow 2$ coding and the $2 \leftarrow 4$ coding.

21. How about a $1 \leftarrow 1$ machine?
    How about a $2 \leftarrow 1$ machine?

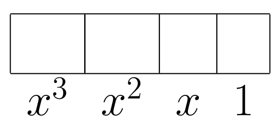
22. Here's a new machine: $1|-1 \leftarrow 0|2$. It takes two dots in one box and converts them into an "anti-dot" and a proper dot. Here are two possible conversions:



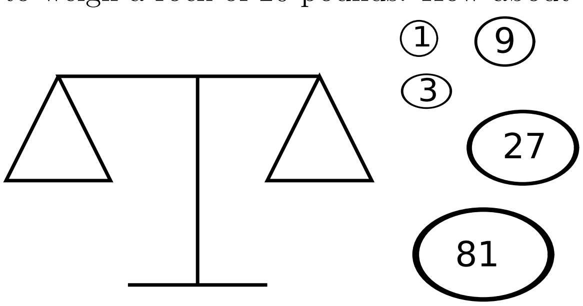
(a) Encode 20 with the $1|-1 \leftarrow 0|2$ machine.

(b) What base 10 number is represented by $1|1|0|-1$?

(c) What is the base of the $1|-1 \leftarrow 0|2$ machine?
The following idea might help (find x):



(d) Suppose you have a simple balance scale and five weights: $1, 3, 9, 27, 81$ pounds. Explain how to weigh a rock of 20 pounds. How about 67 pounds?



Explain what this scale has to do with exploding dots.

Addition, Subtraction and Multiplication

23. Use dot machines to compute:

    (a) $1100_2 + 1101_2$
    (b) $1011_2 - 101_2$
    (c) $100011_2 - 10100_2$
    (d) $101102_3 + 22012_3$
    (e) $10120_3 - 212_3$
    (f) $248_{12} + 9A7_{12}$

    Explain where you see "borrowing".

24. Compute:

    (a) $11121_3 + 120110_3$
    (b) $102_3 \times 201_3$

25. Write down the addition and multiplication tables in bases $2, 3, 4, 5, 12$.

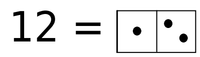
26. Compute (Can you perform these using the same algorithms you learned for base 10?)

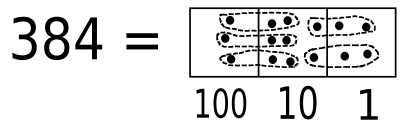
    (a) $1100_2 + 1101_2$
    (b) $1011_2 - 101_2$
    (c) $100011_2 - 10100_2$
    (d) $1011_2 \times 101_2$
    (e) $101102_3 + 22012_3$
    (f) $10120_3 - 212_3$
    (g) $2012_3 \times 112_3$
    (h) $248_{12} + 9A7_{12}$
    (i) $48_{12} \times 55_{12}$

27. Use a $1 \leftarrow 10$ machines to compute 45076×3.

Division!

Here is an example of $384 \div 12$. Notice that this is done by finding groups of 12 (and note below what 12 looks like in boxes).





Thus, you can see $384 \div 12 = 32$.

28. Try this for $387 \div 12$.

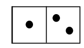
29. Try to use "long division", dot style, using a $1 \leftarrow 10$ machine:
    (a) $235431 \div 101$
    (b) $30176 \div 23$
    (c) $2798 \div 12$

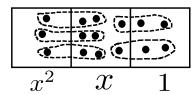
30. (a) Use a $1 \leftarrow 5$ machine to compute (in base 5): $2014_5 \div 11_5$
    (b) Convert this division problem into a problem base 10.

31. Division! (Can you use "long division"?)
    (a) $4210_{10} \div 12_{10}$
    (b) $1000_2 \div 11_2$
    (c) $100_2 \div 10$
    (d) $111_2 \div 11$
    (e) $4210_5 \div 12_5$
    (f) $4210_7 \div 12_7$
    (g) Make up your own problems to divide!

A $1 \leftarrow x$ machine

Here is how we can find $(3x^2 + 8x + 4) \div (x + 2)$: Notice that this is done by finding groups of $x + 2$ (and note below what $x + 2$ looks like in boxes).

$x + 2 =$ 

$3x^2 + 8x + 4 =$ 

Thus, you can see $(3x^2 + 8x + 4) \div (x + 2) = 3x + 2$:

32. Compute $(x^4 + 2x^3 + 4x^2 + 6x + 3) \div (x^2 + 3)$ using a $1 \leftarrow x$ machine.

33. Compute $(x^3 - 3x + 2) \div (x + 2)$ using dots and anti-dots.

34. Compute

    (a) $\dfrac{x^3 - 3x^2 + 3x - 1}{x - 1}$

    (b) $\dfrac{4x^3 - 14x^2 + 14x - 3}{2x - 3}$

    (c) $\dfrac{4x^5 - 2x^4 + 7x^3 - 4x^2 + 6x - 1}{x^2 - x + 1}$

    (d) $\dfrac{x^{10} - 1}{x^2 - 1}$

    (e) Challenge: Is there a way to conduct the dots and boxes approach with ease on paper? Rather than draw boxes and dots, can one work with tables of numbers that keep track of coefficients? (The word "synthetic" is often used for algorithms one creates that are a step or two removed from that actual process at hand.)

More Division

35. Compute

    (a) $\dfrac{x^2 - 1}{x - 1}$

    (b) $\dfrac{x^4 - 1}{x - 1}$

    (c) $\dfrac{x^6 - 1}{x - 1}$

    (d) $\dfrac{x^{\text{even}-1}}{x - 1}$

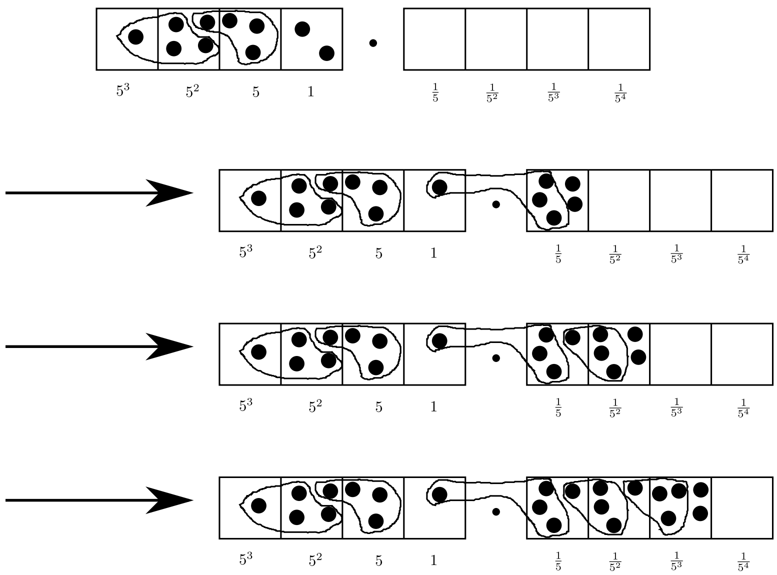
    (e) $\dfrac{x^6 - 1}{x + 1}$

36. (a) Compute $(x^6 + x^5 + 5x^4 + 5x^3 + 9x^2 + 5x + 2) \div (x^2 + x + 2)$

    (b) Put $x = 10$ into your answer for the previous division. What long division arithmetic problem have you solved?

Decimals!

Here is $1432 \div 13$, in base 5, using "unexploding dots" (to get decimals):



Thus, you can see $1432 \div 13 = 110.1111\ldots$.

37. Compute $8 \div 3$ in a base 10 machine (with decimals!).

38. Compute $1 \div 11$ in a base 3 machine.

39. Find the base 4 representation of $\frac{2}{5}$.

40. What fraction has decimal expansion $0.32323232\ldots$ in base 7?

41. Written in base 9, let $x = 0.131313\ldots$. Find the numerator and denominator for x, written as a fraction base 10.

Binary Puzzles

42. Suppose you are a xeno-archeologist who has found an elementary school textbook from an ancient alien civilization. Although most of the book is no longer legible, you have found one equation that says: $3 \times 4 = 10$. How many fingers do you think the aliens have on each hand?

43. Does there exist a number system where the following equations are true simultaneously?

    (a) $3 + 4 = 10$ and $3 \times 4 = 15$?

    (b) $2 + 3 = 5$ and $2 \times 3 = 11$?

44. In base 10, you can tell if a number is even based on whether or not its last digit is even. Find a condition (involving the representation of a number) that allows you to determine whether a number is odd or even

    (a) in the base 3 number system

    (b) in the base n number system

45. A blackboard has a half-erased calculation exercise:

$$\begin{array}{cccccc} & 2 & 3 & ? & 5 & ? \\ + & 1 & ? & 6 & 4 & 2 \\ \hline & 4 & 2 & 4 & 2 & 3 \end{array}$$

    What number system was used and what are the missing digits?

46. A spaceship full of hostile aliens is about to land on Earth. The aliens are very fond of Earth donuts, and you have persuaded them to leave you in peace in exchange for one donut for each alien on board. The captain radios down and says: "There are 100 of us total on board, and we would like 24 jelly donuts and 32 chocolate donuts with sprinkles." What number system are they using?

47. Calculate $71_9 - 61_8 + 51_7 - 41_6 + 31_5 - 21_4$

48. Solve for x and y: $51_x + 71_y = 10^2$