Counting the Steps Needed in the Euclidean Algorithm

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Many thanks to Vincent Kieftenbeld who originally designed this math circle activity.

The Euclidean Algorithm is a way to find the greatest common divisor of two whole numbers. Just to help us remember about the greatest common divisor (gcd), here are a few exercises.

1. Find $\gcd(42, 56)$.
2. Find $\gcd(75, 63)$.
3. Find $\gcd(100, 126)$.

When asked to find the gcd, most students usually factor the integers. This usually works great when the integers are relatively small. But, factoring gets difficult quickly. For example, see how much work do you need to compute the following:

4. Find $\gcd(3400, 4284)$.

Because factoring large numbers is so difficult, it is often a surprise that finding greatest common divisors is actually quite easy, even for large numbers.

In order to use the Euclidean Algorithm one first needs to understand division with remainders. If you want to divide 93 by 5 you would get a quotient of 18 and a remainder of 3:

$$93 = 18 \times 5 + 3$$

We will perform division like this repeated in this math circle activity.

5. Complete the following table. Where in the table can you find $\gcd(56, 42)$?

Step Num	**Dividend**		**Quotient**		**Divisor**		**Remainder**
1	56	=		×	42	+	
2	42	=		×	14	+	

6. Look carefully at the numbers in the table above. Do you see a pattern? Use this pattern to complete the following table. What is gcd(13, 8) and where is it found in the table below.

Step Num	**Dividend**		**Quotient**		**Divisor**		**Remainder**
1	13	=	1	×	8	+	5
		=		×		+	
		=		×		+	
		=		×		+	
		=		×		+	

The method of finding the gcd of two numbers using a table as in Problems 5 and 6 is called the Euclidean Algorithm. Most of the time when studying the Euclidean Algorithm you are interested in either finding the greatest common divisor or something similar. If you want, you can go ahead and practice using the Euclidean Algorithm on Problems 1, 2, and 3.

For this activity, we are going to investigate the number of steps needed to find the gcd. Notice that we when found $\gcd(56, 42)$, we needed 2 steps in our table. But when we found $\gcd(13, 8)$, we needed 5 steps in our table.

7. Record the number of steps required for the Euclidean Algorithm in the "Recording Triangle." For example, $\gcd(13, 8)$ took 5 steps so you should record a 5 in row 13 and column 8.

8. How many rows do you need to find $\gcd(4, 1)$ or $\gcd(17, 1)$ using the Euclidean Algorithm? Fill out the values for $\gcd(a, 1)$ in the Recording Triangle.

9. Determine the number of steps to find $\gcd(4, 2)$ and $\gcd(18, 9)$.

10. When b divides a, how many step do you need? Without actually finding the gcd's, you can fill out all the cells in the Recording Triangle column with even numbers.

11. Fill out the entries in the Recording Triangle for the multiples of 3 in the third column, for the multiples of 4 in the fourth column, and so on.

12. The first open cell in the Recording Triangle is for $a = 3$ and $b = 2$. How many steps do you need to find $\gcd(3, 2) = 1$ using the Euclidean Algorithm? What about finding $\gcd(4, 3)$?

13. Fill out all entries in the Recording Triangle along the diagonal for

$$\gcd(3, 2), \gcd(4, 3), \gcd(5, 4), \ldots, \gcd(20, 19)$$

14. Use the Euclidean Algorithm to find $\gcd(5, 2)$ and $\gcd(7, 2)$. What is the pattern? Use this to complete the whole second column in the Recording Triangle.

15. It looks like a 1 in the Recording Triangle is always followed by a 2. Is this always the case? How many steps do you need to find $\gcd(5, 3)$?

16. Dividing 5 by 3 leaves a remainder of 2. Dividing 8 by 3 also leaves a reminder of 2. If you need 3 steps to find $\gcd(5, 3)$, how many step do you need to find $\gcd(8, 3)$?

Think about this: 5 and 8 both have a remainder of 2 when you divide by 3. You also need the same number of steps in the Euclidean Algorithm to find $\gcd(5, 3)$ and $\gcd(8, 3)$. This is in fact a very general pattern: When x and y have the same remainder upon division by b, you need the same number of steps to find $\gcd(x, b)$ and $\gcd(y, b)$.

17. Use the pattern described above to complete as many entries in the Recording Triangle as possible.

18. There are more difficult patterns for you to find in the table. For example, compare the number of steps you need to find $\gcd(3, 2)$, $\gcd(6, 4)$, and $\gcd(12, 8)$. What do you notice about the rows? What do 3/2, 6/4, and 12/8 have in common? Now look at the entries for $\gcd(5, 3)$, $\gcd(10, 6)$, and $\gcd(20, 12)$.

19. Complete the Recording Table using the patterns that you have found. You may need to figure out some of the entries by actually computing the gcd's, but try to do that as little as possible.

20. What is the first pair of numbers that takes one step? Which pair is the smallest that takes two steps?

21. Going from the upper left to the lower right, circle or color the first entry for 1, 2, 3, ... in the Recording Triangle. Fille out the table:

Rows Needed	**a**	**b**
1		
2		
3		
4		
5		

22. Make a guess for the smallest pair that will require 6 steps to find the gcd. Check your guess by actually finding the gcd for these numbers.

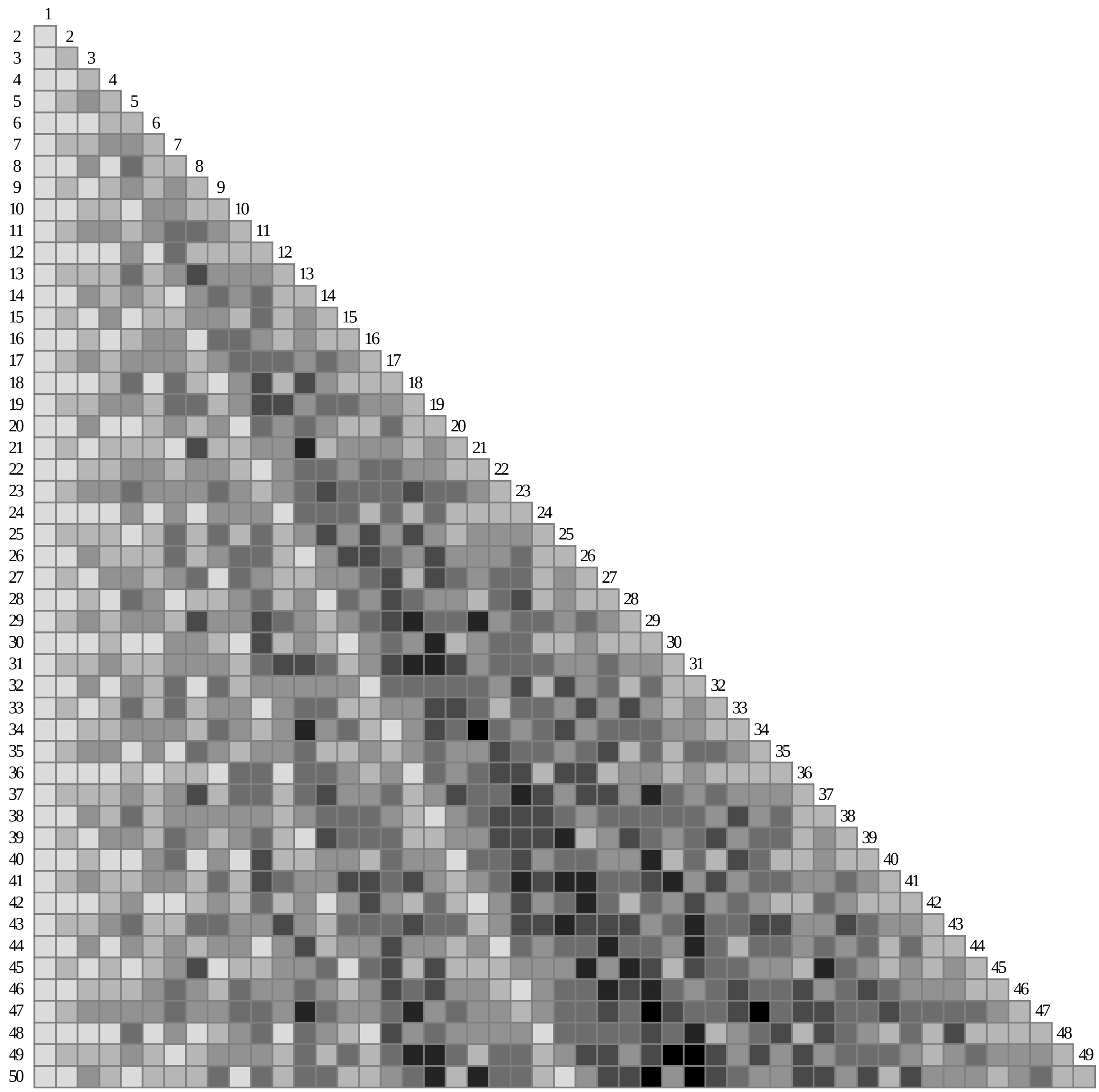
23. The Fibonacci numbers are the numbers

$$1, 1, 2, 3, 5, 8, 13, \ldots$$

What is the next Fibonacci number? What is the pattern?

24. Consider the pair of successive Fibonacci numbers $(5, 3)$. Where in the Recording Triangle can you find this pair? What is special about that entry?

25. What about other pairs of successive Fibonacci numbers? Make a conjecture about the connection between Fibonacci numbers and the number of steps needed to find their gcd.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2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50

a \ *b*	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
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