Math 233 - October 26, 2009
Solutions

- What is so great about $G = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$?
- How can we extend the idea of integration to $\mathbb{R}^2$?

1. True/False. If false, determine how the question can be made true.

   (a) The work done by a vector field around a loop is zero.
   Solution: False. True if vector field is conservative.

   (b) If $P, Q \in \mathbb{R}^n$ then the work that vector field F does from P to Q is the negative of the work done from Q to P.
   Solution: False. True if the vector field is conservative or if the paths traced are the same (but in opposite directions).

   (c) Suppose F is not conservative and $r(t)$ a path from P to Q. Then the work that F does along r from P to Q is equal to the negative of the work done along the reverse path from Q to P.
   Solution: True

   (d) Suppose the curl of the vector field F is equal to $(0, 0, 0)$. Then the word done around every loop is equal to 0.
   Solution: False, not true for the the vector field $G = (-y/(x^2 + y^2), x/(x^2 + y^2))$.

2. Find the line integrals using potential functions.

   (a) Let $r(t)$ be the line from $(1, 5)$ to $(-2, 3)$.
   $\int_r y^2\, dx + (2xy - 1)\, dy = \; -21 - 20 = -41$

   (b) Let $r(t) = (1 + \cos t, -2 + 3\sin t)$ for $0 \leq t \leq \pi/2$.
   $\int_r (2xy^3 + y - 1)\, dx + (3x^2y^2 + x)\, dy = 1 - (-38) = 39$

   (c) Let $r(t)$ be the upper hemisphere of the unit circle oriented counter-clockwise.
   $\int_r (y - e^y)\, dx + (x - xe^y)\, dy = 1 - (-1) = 2$

Lecture Problems

3. Let $r(t)$ be the unit circle oriented positively. Compute

$$\frac{1}{2\pi} \oint_r \frac{y^3 + x^2y - 4x}{2(y^2 + x^2)^2}\, dx - \frac{(xy^2 + 4y + x^3)}{2(y^2 + x^2)^2}\, dy$$
$$= \frac{1}{2\pi} \int_0^{2\pi} -\frac{1}{2}\, dt = -\frac{1}{2}$$

4. Let

$$G = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$
$$F = \left(\frac{y^3+x^2y-4x}{2(y^2+x^2)^2}, -\frac{(xy^2+4y+x^3)}{2(y^2+x^2)^2}\ dy \right)$$

Find k and ϕ so that $F = kG + \nabla\phi$.

$$k = -\frac{1}{2}$$
$$\nabla\phi = \left(-\frac{2x}{(y^2+x^2)^2}, -\frac{2y}{(y^2+x^2)^2} \right)$$
$$\phi = \frac{1}{y^2+x^2}$$