Math 233 - October 20, 2009
Solutions

- Test Review!

1. Find the maximum rate of change among all possible directions for the function $f(x, y) = x^2y^3 + 2x^4y$ at the point $(-1, 1)$.

   Solution: $5\sqrt{5}$

2. Let $f(x, y) = \ln(x^2 + y^3)$.

   (a) Find a linearization of f near the point $(2, 1)$
   Solution: $L(x, y) = \ln 5 + \frac{4}{5}(x - 2) + \frac{3}{5}(y - 1)$

   (b) Find the equation of the tangent plane to the graph at $(2, 1, \ln 5)$.
   Solution: Same as previous, $z = L(x, y)$.

   (c) Find a unit vector in the direction in which f increases most rapidly from the point $(2, 1)$.
   Solution: $\frac{1}{5}(4, 3)$.

3. Approximate $f(8.04, 11.97)$ for a function $f(x, y)$ for which you know $f(8, 12) = 3$, $f_x(8, 12) = -1$ and $f_y(8, 12) = 2$.

   Solution: 2.90

4. Find the equation of the tangent plane at $(2, 1, 3)$ to $z = \sqrt{20 - x^2 - 7y^2}$.

   Solution: $z = 3 - \frac{2}{3}(x - 2) - \frac{7}{3}(y - 1)$

5. find the directional derivative of $f(x, y) = x^2 + xy$ at $(1, 2)$ in the direction of the vector $(1, 1)$

   Solution: $5/\sqrt{2}$

6. Find the equation of the tangent plane at the point $(1, 2, 0)$ to the surface $x^3 + 2y^2 + z^2 - 2xy - 2yz = 5$.

   Solution: $-(x - 1) + 6(y - 2) - 4z = 0$.

7. Let $f(x, y)$ be differentiable and defined on the set $A = \{(x, y)|x^2+4y^2 < 1\}$. Suppose $f_x(0, 0) = 0$ and $f_y(0, 0) = 0$. Which must be true

   (a) f has a local maximum on A

   (b) f has a local minimum on A

   (c) f has a local extremum on A

   (d) f has a absolute maximum on A

   (e) f has a absolute minimum on A

   Solution: None must be true

8. Find and classify all critical points to $f(x, y) = 8y^3 + 12x^2 - 24xy$.

   Solution: Saddle at $(0, 0$, local min at $(1, 1)$.

9. Find the max of $f(x, y) = 3y + 4x$ on the circle $x^2 + y^2 = 1$.

   Solution: 5

10. Find the maximum rate of change of $f(x, y) = x^3y^4$ at the point $(1, 1)$.

    Solution: 5

11. Find the maximum of $f(x, y, z) = 2x + y + z$ for points satisfying $x^2 + y^2 + z^2 \leq 6$.

    Solution: 6

12. Find the minimum of $f(x, y) = 2x^2 + 3y^2$ subject to $x^2 + y^2 = 4$.

    Solution: 8

13. Find and classify all critical points of $f(x, y) = x^3 + 3xy + 3y^2/2$

    Solution: Saddle at $(0, 0)$ and local min at $(1, -1)$.

14. Let $f(x, y) = xy$. Find all extrema on the set $4x^2 + 9y^2 \leq 36$.

    Solution: Saddle at $(0, 0)$, max of 3, min of -3 (max/min on boundary).

15. Find and classify all critical points of $f(x, y) = 3xy^2 + x^3 - 3y^2 - 3x^2 + 5$.

    Solution: Local max at $(0, 0)$, saddle at $(1, -1)$. Other critical points (that also need classified): $(2, 0)$ and $(1, 1)$.

16. Let a, b, c be fixed positive numbers. Find the positive numbers x, y, z such that $ax+by+cz = 99$ and xyz is maximal. (You may be able to do this without calculus, but obviously you will need to know how to do it using calculus.)

    Solution: $x = 33/a$, $y = 33/b$, $z = 33/c$.

17. Let $f(x, y) = x/y^2$. Make two approximations of $f(3.2, 1.1)$. First use the linerization at the point $(4, 1)$ and then use the second order Taylor polynomial at the point $(4, 1)$.

    Solution: $L(3.2, 1.1) = 2.4$ $T_2(3.2, 1.1) = 2.68$

    You should, of course, compare this to $f(3.2, 1.1) = 2.6446$

18. Let $f(x, y) = x^3 + y^4 - \ln(x + y - 2)$. Make two approximations of $f(1.2, 1.9)$. First use the linerization at the point $(1, 2)$ and then use the second order Taylor polynomial at the point $(1, 2)$.

    Solution: $L(1.2, 1.9) = 14.3$ $T_2(1.2, 1.9) = 14.665$

    You should, of course, compare this to $f(1.2, 1.9) = 14.6647898$

19. True/False

(a) The set in $\mathbb{R}^2$ defined by $xy = 1$ is closed.
Solution: True

(b) The set in $\mathbb{R}^2$ defined by $xy = 1$ is bounded.
Solution: False

(c) If f is differentiable then the tangent plane at any critical is parallel to the xy plane.
Solution: True

(d) A continuous function on a closed domain has a maximum and minimum on that closed set.
Solution: False

(e) A continuous function on a bounded domain has a maximum and minimum on that bounded set.
Solution: False

(f) All vector fields have potential functions.
Solution: False

(g) All gradient vector fields have potential functions.
Solution: True

(h) If the curl of a vector field is $(0,0,0)$ then the vector field has a potential function.
Solution: False (but almost true)

20. Find the curl of $F = (e^x \sin y, e^x \cos y, z)$

    Solution: $(0,0,0)$.

21. Let $f(x,y,z) = x^2 \cos(z/x) + y/\ln(x - \tan z)$.

    Find a potential function for the vector field $F(x,y) = \nabla f(x,y)$.

    Solution: f is the potential function.

22. Let $F = (2xy, x^2 + 1)$. Find a potential function $f(x,y)$ for F and find $f(1,1) - f(0,0)$.

    Solution: 2

23. Let $F = (e^y, xe^y, (z+1)e^z)$. Let f be a potential function for F and find $f(1,1,1) - f(0,0,0)$.

    Solution: $2e$

24. Find the curl of the vector field $F = (z^2, zy, x^2 - z)$

    Solution: $(-y, 2z - 2x, 0)$

25. Find the absolute maximum of $f(x,y) = x^2 - 2xy + 2y$ on the rectangle $\{(x,y)|0 \leq x \leq 3 \text{ and } 0 \leq y \leq 2\}$.

    Solution: 9.

26. Find and classify all criticals points of the function $f(x,y) = x^4 + y^4 - 4xy + 2$.

    Solution: Saddle at $(0,0)$, local max at $(1,1)$, local max at $(-1,-1)$.

27. Use Lagrange multipliers to find the minimum distance from the curve $y = x^2 - 1$ to the origin.

    Solution: Minimum occurs at $x = \pm 1/\sqrt{2}$.

28. Find a potential function for the vector field $F = (x, y + z, z^2)$.

    Solution: F is not conservative.

29. Find a potential function for the vector field $F = (e^x, 2yz, y^2)$.

    Solution: $f = e^x + y^2 z$

30. Find a potential function for the vector field $F = (ye^z, ze^y, xe^z)$.

    Solution: F is not conservative.

31. Find the divergence of $F = (5x^2 + 3yz, 7y^2 + 2xz, 3z^2 + 3xy)$

    Solution: $10x + 14y + 6z$.