Math 233 - October 20, 2009

- Test Review!

1. Find the maximum rate of change among all possible directions for the function $f(x, y) = x^2y^3 + 2x^4y$ at the point $(-1, 1)$.
2. Let $f(x, y) = \ln(x^2 + y^3)$.
   (a) Find a linearization of f near the point $(2, 1)$
   (b) Find the equation of the tangent plane to the graph at $(2, 1, \ln 5)$.
   (c) Find a unit vector in the direction in which f increases most rapidly from the point $(2, 1)$.

1. Find the maximum rate of change among all possible directions for the function $f(x, y) = x^2y^3 + 2x^4y$ at the point $(-1, 1)$.
   Solution: $5\sqrt{5}$
2. Let $f(x, y) = \ln(x^2 + y^3)$.
   (a) Find a linearization of f near the point $(2, 1)$
   Solution: $L(x, y) = \ln 5 + \frac{4}{5}(x - 2) + \frac{3}{5}(y - 1)$
   (b) Find the equation of the tangent plane to the graph at $(2, 1, \ln 5)$.
   Solution: Same as previous, $z = L(x, y)$.
   (c) Find a unit vector in the direction in which f increases most rapidly from the point $(2, 1)$.
   Solution: $\frac{1}{5}(4, 3)$.

3. Approximate $f(8.04, 11.97)$ for a function $f(x, y)$ for which you know $f(8, 12) = 3$, $f_x(8, 12) = -1$ and $f_y(8, 12) = 2$.
4. Find the equation of the tangent plane at $(2, 1, 3)$ to $z = \sqrt{20 - x^2 - 7y^2}$.
5. find the directional derivative of $f(x, y) = x^2 + xy$ at $(1, 2)$ in the direction of the vector $(1, 1)$
6. Find the equation of the tangent plane at the point $(1, 2, 0)$ to the surface $x^3 + 2y^2 + z^2 - 2xy - 2yz = 5$.

3. Approximate $f(8.04, 11.97)$ for a function $f(x, y)$ for which you know $f(8, 12) = 3$, $f_x(8, 12) = -1$ and $f_y(8, 12) = 2$.
   Solution: 2.90
4. Find the equation of the tangent plane at $(2, 1, 3)$ to $z = \sqrt{20 - x^2 - 7y^2}$.
   Solution: $z = 3 - \frac{2}{3}(x - 2) - \frac{7}{3}(y - 1)$
5. find the directional derivative of $f(x, y) = x^2 + xy$ at $(1, 2)$ in the direction of the vector $(1, 1)$
   Solution: $5/\sqrt{2}$
6. Find the equation of the tangent plane at the point $(1, 2, 0)$ to the surface $x^3 + 2y^2 + z^2 - 2xy - 2yz = 5$.
   Solution: $-(x - 1) + 6(y - 2) - 4z = 0$.

7. Let $f(x, y)$ be differentiable and defined on the set $A = \{(x, y) | x^2 + 4y^2 < 1\}$. Suppose $f_x(0, 0) = 0$ and $f_y(0, 0) = 0$. Which must be true
   (a) f has a local maximum on A
   (b) f has a local minimum on A
   (c) f has a local extremum on A
   (d) f has a absolute maximum on A
   (e) f has a absolute minimum on A
8. Find and classify all critical points to $f(x, y) = 8y^3 + 12x^2 - 24xy$.
9. Find the max of $f(x, y) = 3y + 4x$ on the circle $x^2 + y^2 = 1$.
10. Find the maximum rate of change of $f(x, y) = x^3y^4$ at the point $(1, 1)$.
11. Find the maximum of $f(x, y, z) = 2x + y + z$ for points satisfying $x^2 + y^2 + z^2 \leq 6$.

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   (a) f has a local maximum on A
   (b) f has a local minimum on A
   (c) f has a local extremum on A
   (d) f has a absolute maximum on A
   (e) f has a absolute minimum on A

   Solution: None must be true
8. Find and classify all critical points to $f(x, y) = 8y^3 + 12x^2 - 24xy$.
   Solution: Saddle at $(0, 0$, local min at $(1, 1)$.
9. Find the max of $f(x, y) = 3y + 4x$ on the circle $x^2 + y^2 = 1$.
   Solution: 5
10. Find the maximum rate of change of $f(x, y) = x^3y^4$ at the point $(1, 1)$.
    Solution: 5
11. Find the maximum of $f(x, y, z) = 2x + y + z$ for points satisfying $x^2 + y^2 + z^2 \leq 6$.
    Solution: 6

12. Find the minimum of $f(x,y) = 2x^2 + 3y^2$ subject to $x^2 + y^2 = 4$.
13. Find and classify all critical points of $f(x,y) = x^3 + 3xy + 3y^2/2$
14. Let $f(x,y) = xy$. Find all extrema on the set $4x^2 + 9y^2 \leq 36$.
15. Find and classify all critical points of
    $f(x,y) = 3xy^2 + x^3 - 3y^2 - 3x^2 + 5$.
16. Let a, b, c be fixed positive numbers. Find the positive numbers x, y, z such that $ax + by + cz = 99$ and xyz is maximal. (You may be able to do this without calculus, but obviously you will need to know how to do it using calculus.)

12. Find the minimum of $f(x,y) = 2x^2 + 3y^2$ subject to $x^2 + y^2 = 4$.
Solution: 8
13. Find and classify all critical points of $f(x,y) = x^3 + 3xy + 3y^2/2$
Solution: Saddle at $(0,0)$ and local min at $(1,-1)$.
14. Let $f(x,y) = xy$. Find all extrema on the set $4x^2 + 9y^2 \leq 36$.
Solution: Saddle at $(0,0)$, max of 3, min of -3 (max/min on boundary).
15. Find and classify all critical points of $f(x,y) = 3xy^2 + x^3 - 3y^2 - 3x^2 + 5$.
Solution: Local max at $(0,0)$, saddle at $(1,-1)$. Other critical points (that also need classified): $(2,0)$ and $(1,1)$.
16. Let a, b, c be fixed positive numbers. Find the positive numbers x, y, z such that $ax + by + cz = 99$ and xyz is maximal. (You may be able to do this without calculus, but obviously you will need to know how to do it using calculus.)
Solution: $x = 33/a$, $y = 33/b$, $z = 33/c$.

17. Let $f(x, y) = x/y^2$. Make two approximations of $f(3.2, 1.1)$. First use the linerization at the point $(4, 1)$ and then use the second order Taylor polynomial at the point $(4, 1)$.
18. Let $f(x, y) = x^3 + y^4 - \ln(x + y - 2)$. Make two approximations of $f(1.2, 1.9)$. First use the linerization at the point $(1, 2)$ and then use the second order Taylor polynomial at the point $(1, 2)$.

17. Let $f(x,y) = x/y^2$. Make two approximations of $f(3.2, 1.1)$. First use the linerization at the point $(4,1)$ and then use the second order Taylor polynomial at the point $(4,1)$.
Solution: $L(3.2, 1.1) = 2.4$ $T_2(3.2, 1.1) = 2.68$
You should, of course, compare this to $f(3.2, 1.1) = 2.6446$
18. Let $f(x,y) = x^3 + y^4 - \ln(x + y - 2)$. Make two approximations of $f(1.2, 1.9)$. First use the linerization at the point $(1,2)$ and then use the second order Taylor polynomial at the point $(1,2)$.
Solution: $L(1.2, 1.9) = 14.3$ $T_2(1.2, 1.9) = 14.665$
You should, of course, compare this to $f(1.2, 1.9) = 14.6647898$

19. True/False

(a) The set in $\mathbb{R}^2$ defined by $xy = 1$ is closed.
(b) The set in $\mathbb{R}^2$ defined by $xy = 1$ is bounded.
(c) If f is differentiable then the tangent plane at any critical is parallel to the xy plane.
(d) A continuous function on a closed domain has a maximum and minimum on that closed set.
(e) A continuous function on a bounded domain has a maximum and minimum on that bounded set.
(f) All vector fields have potential functions.
(g) All gradient vector fields have potential functions.
(h) If the curl of a vector field is $(0, 0, 0)$ then the vector field has a potential function.

19. True/False

(a) The set in $\mathbb{R}^2$ defined by $xy = 1$ is closed.
Solution: True

(b) The set in $\mathbb{R}^2$ defined by $xy = 1$ is bounded.
Solution: False

(c) If f is differentiable then the tangent plane at any critical is parallel to the xy plane.
Solution: True

(d) A continuous function on a closed domain has a maximum and minimum on that closed set.
Solution: False

(e) A continuous function on a bounded domain has a maximum and minimum on that bounded set.
Solution: False

(f) All vector fields have potential functions.
Solution: False

(g) All gradient vector fields have potential functions.
Solution: True

(h) If the curl of a vector field is $(0, 0, 0)$ then the vector field has a potential function.
Solution: False (but almost true)

20. Find the curl of $F = (e^x \sin y, e^x \cos y, z)$
21. Let $f(x, y, z) = x^2 \cos(z/x) + y/\ln(x - \tan z)$.
    Find a potential function for the vector field $F(x, y) = \nabla f(x, y)$.
22. Let $F = (2xy, x^2 + 1)$. Find a potential function $f(x, y)$ for F and find $f(1, 1) - f(0, 0)$.
23. Let $F = (e^y, xe^y, (z + 1)e^z)$. Let f be a potential function for F and find $f(1, 1, 1) - f(0, 0, 0)$.
24. Find the curl of the vector field $F = (z^2, zy, x^2 - z)$

20. Find the curl of $F = (e^x \sin y, e^x \cos y, z)$
    Solution: $(0, 0, 0)$.
21. Let $f(x, y, z) = x^2 \cos(z/x) + y/\ln(x - \tan z)$.
    Find a potential function for the vector field $F(x, y) = \nabla f(x, y)$.
    Solution: f is the potential function.
22. Let $F = (2xy, x^2 + 1)$. Find a potential function $f(x, y)$ for F and find $f(1, 1) - f(0, 0)$.
    Solution: 2
23. Let $F = (e^y, xe^y, (z + 1)e^z)$. Let f be a potential function for F and find $f(1, 1, 1) - f(0, 0, 0)$.
    Solution: $2e$
24. Find the curl of the vector field $F = (z^2, zy, x^2 - z)$
    Solution: $(-y, 2z - 2x, 0)$

25. Find the absolute maximum of $f(x,y) = x^2 - 2xy + 2y$ on the rectangle $\{(x,y)|0 \leq x \leq 3 \text{ and } 0 \leq y \leq 2\}$.
26. Find and classify all criticals points of the function $f(x,y) = x^4 + y^4 - 4xy + 2$.
27. Use Lagrange multipliers to find the minimum distance from the curve $y = x^2 - 1$ to the origin.

25. Find the absolute maximum of $f(x,y) = x^2 - 2xy + 2y$ on the rectangle $\{(x,y)|0 \leq x \leq 3 \text{ and } 0 \leq y \leq 2\}$.
Solution: 9.

26. Find and classify all criticals points of the function $f(x,y) = x^4 + y^4 - 4xy + 2$.
Solution: Saddle at $(0,0)$, local max at $(1,1)$, local max at $(-1,-1)$.

27. Use Lagrange multipliers to find the minimum distance from the curve $y = x^2 - 1$ to the origin.
Solution: Minimum occurs at $x = \pm 1/\sqrt{2}$.

28. Find a potential function for the vector field $F = (x, y + z, z^2)$.
29. Find a potential function for the vector field $F = (e^x, 2yz, y^2)$.
30. Find a potential function for the vector field $F = (ye^z, ze^y, xe^z)$.
31. Find the divergence of $F = (5x^2 + 3yz, 7y^2 + 2xz, 3z^2 + 3xy)$

28. Find a potential function for the vector field $F = (x, y + z, z^2)$.
Solution: F is not conservative.

29. Find a potential function for the vector field $F = (e^x, 2yz, y^2)$.
Solution: $f = e^x + y^2 z$

30. Find a potential function for the vector field $F = (ye^z, ze^y, xe^z)$.
Solution: F is not conservative.

31. Find the divergence of $F = (5x^2 + 3yz, 7y^2 + 2xz, 3z^2 + 3xy)$
Solution: $10x + 14y + 6z$.