Exam 2: Oct 21, Sections 4.2 - 7.3

- Tangent planes
- Directional derivative
- Critical points
- Extrema on a closed and bounded domain
- Lagrange multipliers
- Taylor polynomials (up to second order)
- Second derivative test for extrema
- Potential functions: existence, finding
\[ G = \left( -\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right) \]

A potential function for \( G \) in the right half-plane \((x > 0)\) is

\[ \phi = \arctan \frac{y}{x} \]

Can we extend this potential function to be defined continuously on \( \mathbb{R}^2 \)?
1. Find $\nabla \phi$ for

$$\phi = \arccos \frac{x}{\sqrt{x^2 + y^2}}$$

Two hints: $\frac{d}{dt} \arccos t = -\frac{1}{\sqrt{1-t^2}}$

and remember that $\sqrt{y^2} = |y|$ and not $y$. 
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$$\nabla \phi = \left( -\frac{y^2}{|y|(x^2 + y^2)}, \frac{xy}{|y|(x^2 + y^2)} \right)$$

Thus, for $y \geq 0$ we have $\phi$ is a potential function for $G$. 
Lecture Problems

2. For each vector field, determine the domain, find a potential function or explain why a potential function does not exist.

(a) \( F = (1/(x - y), 1/(y - x)) \)
\( \phi = \)
(b) \( F = (-2x/(x^2 + y^2)^2, -2y/(x^2 + y^2)) \)
\( \phi = \)
(c) \( F = (1/y, -x/y^2) \)
\( \phi = \)
(d) \( F = (-1/(x^2 y), -1/xy^2) \)
\( \phi = \)
(e) \( F(x, y, z) = (y, x + x/(y^2 + z^2), -y/(y^2 + z^2)) \)
\( \phi = \)
(f) \( F(x, y, z, w) = (-y/(x^2 + y^2), x/(x^2 + y^2), 1/w^2, -2z/w^3) \)
\( \phi = \)
Lecture Problems

2. For each vector field, determine the domain, find a potential function or explain why a potential function does not exist.

(a) \( F = \left( \frac{1}{x - y}, \frac{1}{y - x} \right) \)
\[ \phi = \ln |x - y| \]

(b) \( F = \left( \frac{-2x}{x^2 + y^2}, \frac{-2y}{x^2 + y^2} \right) \)
\[ \phi = \frac{1}{x^2 + y^2} \]

(c) \( F = \left( \frac{1}{y}, -\frac{x}{y^2} \right) \)
\[ \phi = \frac{x}{y} \]

(d) \( F = \left( \frac{-1}{x^2 y}, \frac{-1}{xy^2} \right) \)
\[ \phi = \frac{1}{xy} \]

(e) \( F(x, y, z) = \left( y, x + x/(y^2 + z^2), -y/(y^2 + z^2) \right) \)
\[ \phi = xy + \arctan(y/z) \]

(f) \( F(x, y, z, w) = \left( -\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, \frac{1}{w^2}, -\frac{2z}{w^3} \right) \)
\[ \phi = \frac{z}{w^2} + \arctan(y/x) \]
3. Find the line integrals
   
   (a) $F = (x + y, x^2)$, $r(t) = (t, t^2)$ for $0 \leq t \leq 1$.
       $$\int_r F \cdot dr =$$

   (b) $F = (-y, x)$, $r(t) = (t, t^2)$ for $0 \leq t \leq 1$.
       $$\int_r F \cdot dr =$$

   (c) $F = (-y^2, x)$, $r(t) = (t + 1, 2t)$ for $-2 \leq t \leq 2$.
       $$\int_r F \cdot dr =$$
3. Find the line integrals

(a) $F = (x + y, x^2)$, $r(t) = (t, t^2)$ for $0 \leq t \leq 1$.

\[ \int_r F \cdot dr = \frac{4}{3} \]

(b) $F = (-y, x)$, $r(t) = (t, t^2)$ for $0 \leq t \leq 1$.

\[ \int_r F \cdot dr = \frac{4}{3} \]

(c) $F = (-y^2, x)$, $r(t) = (t + 1, 2t)$ for $-2 \leq t \leq 2$.

\[ \int_r F \cdot dr = -\frac{40}{3} \]