Math 233 - October 12, 2009
Solutions

- Do all vector fields have potential functions?
- How can you tell if a vector field has a potential function?

1. Compute (Hint, use the substitution \( u = y/x \))
   \[
   \int \frac{x \, dy}{x^2 + y^2} = \arctan \left( \frac{y}{x} \right) + C
   \]

2. Let \( F = (f_1, f_2, f_3) \) be a vector field where \( f_1, f_2 \) and \( f_3 \) are arbitrary functions. What is
   \[
   \text{curl} F = \nabla \times F = \begin{vmatrix}
   E_1 & E_2 & E_3 \\
   \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
   f_1 & f_2 & f_3
   \end{vmatrix}
   
   = \left( \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right)
   \]

3. For the following vector fields, compute the curl of the vector field and then, if possible, find a potential function for the vector field.
   
   (a) \( F = (yz, xz, xy) \), curl \( F = (0, 0, 0) \)
   \[ \phi = xyz \]

   (b) \( F = (x, y, x) \), curl \( F = (0, -1, 0) \)
   \[ \phi = \text{DNE} \]

   (c) \( F = (2x, 2y, 1) \), curl \( F = (0, 0, 0) \)
   \[ \phi = x^2 + y^2 + z \]

   (d) \( F = (-y, z, x) \), curl \( F = (-1, -1, 1) \)
   \[ \phi = \text{DNE} \]

   (e) \( F = (y, x - 2z, 2z - 2y) \), curl \( F = (0, 0, 0) \)
   \[ \phi = xy - 2yz + z^2 \]

4. Compute the curl of the vector field
   \[
   F = \left( -\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, 0 \right)
   \]
   Solution: This is just a computation and should give curl \( F = (0, 0, 0) \).

Lecture Problems

5. Determine if the following sets are connected
(a) \( A = \{(x, y, z)|x^2 + y^2 \leq 1\} \subset \mathbb{R}^3 \)

Solution: \( A \) is connected.

(b) \( A = \{(x, y, z)|x^2 + y^2 = 1\} \subset \mathbb{R}^3 \)

Solution: \( A \) is connected.

(c) \( A = \{(x, y, z)|x \neq 0\} \subset \mathbb{R}^3 \)

Solution: \( A \) is not connected.

(d) \( A = \{(x, y)|x \neq 0\} \subset \mathbb{R}^2 \)

Solution: \( A \) is not connected.

(e) \( A = \{(x, y, z)|x \neq 0\} \subset \mathbb{R}^3 \)

Solution: \( A \) is not connected.

(f) \( A = \{(x, y)|x \neq 0 \text{ and } y \neq 0\} \subset \mathbb{R}^2 \)

Solution: \( A \) is connected.

(g) \( A = \{(x, y, z)|x \neq 0 \text{ and } y \neq 0\} \subset \mathbb{R}^3 \)

Solution: \( A \) is connected.