Math 233 - October 6, 2009
Solutions

- Understand the nature of critical points for quadratic functions.
- Understand the second derivative test for functions of two variables.

1. Let $f(x, y) = 3x^3 + y^2 - 9x + 4y$.
   (a) Find all critical points for $f(x, y)$.
   Solution: $(1, -2)$ and $(-1, -2)$
   (b) Find the second degree Taylor polynomial at each of the critical points
   Solution: At $(1, -2)$:
   $$T_2(x, y) = -10 + \frac{1}{2}\left(18(x-1)^2 + 2(y+2)^2\right)$$
   At $(-1, -2)$:
   $$T_2(x, y) = 2 + \frac{1}{2}\left(-18(x+1)^2 + 2(y+2)^2\right)$$
   (c) Determine the nature of the two critical points.
   Solution: Local min at $(1, -2)$, saddle at $(-1, -2)$.

Lecture Problems

2. Each of the functions below have a critical point at $(0, 0)$ (why?). Determine if there is a local max, local min or saddle at $(0, 0)$.
   (a) $f(x, y) = 4x^2 + 15y^2$.
   Solution: Min at $(0, 0)$.
   (b) $f(x, y) = 2x^2 + 91y^2$.
   Solution: Min at $(0, 0)$.
   (c) $f(x, y) = -2x^2 - 91y^2$.
   Solution: Max at $(0, 0)$.
   (d) $f(x, y) = -12x^2 - 5y^2$.
   Solution: Max at $(0, 0)$.
   (e) $f(x, y) = x^2 - y^2$.
   Solution: Saddle at $(0, 0)$.
   (f) $f(x, y) = -x^2 + y^2$.
   Solution: Saddle at $(0, 0)$.

3. Starting with the function $f(x, y) = ax^2 + bxy + cy^2$ make the substitution $x = u - \frac{b}{2a}v$ and $y = v$ to transform the functions to functions of u and v. (Simplify your result.)

(a) $f(x,y) = x^2 + xy + y^2$, $f(u,v) = u^2 + \frac{3}{4}v^2$.
(b) $f(x,y) = 2x^2 + xy + y^2$, $f(u,v) = 2u^2 + \frac{7}{8}v^2$.
(c) $f(x,y) = 3x^2 + xy + y^2$, $f(u,v) = 3u^2 + \frac{11}{12}v^2$.
(d) $f(x,y) = x^2 + 2xy + y^2$, $f(u,v) = u^2$.
(e) $f(x,y) = x^2 + 3xy + y^2$, $f(u,v) = u^2 - \frac{5}{4}v^2$.
(f) $f(x,y) = x^2 + xy - 2y^2$, $f(u,v) = u^2 - \frac{9}{4}v^2$.
(g) $f(x,y) = -3x^2 + xy - 5y^2$, $f(u,v) = \, - 3u^2 - \frac{39}{12}v^2$.

4. Determine if the functions in the previous problem have a local max, local min or saddle at $(0,0)$.

(a) $f(x,y) = x^2 + xy + y^2$, **Solution:** Local Min
(b) $f(x,y) = 2x^2 + xy + y^2$, **Solution:** Local Min
(c) $f(x,y) = 3x^2 + xy + y^2$, **Solution:** Local Min
(d) $f(x,y) = x^2 + 2xy + y^2$, **Solution:** Local Min
(e) $f(x,y) = x^2 + 3xy + y^2$, **Solution:** Saddle
(f) $f(x,y) = x^2 + xy - 2y^2$, **Solution:** Saddle
(g) $f(x,y) = -3x^2 + xy - 5y^2$, **Solution:** Local Max

5. Let $f(x,y) = ax^2 + bxy + cy^2$. Assume $a \neq 0$.

(a) Let $x = u - \frac{b}{2a}v$ and $y = v$.

$$f(u,v) = au^2 - \frac{(b^2 - 4ac)}{4a}v^2$$

(b) What conditions will ensure that $f(u,v)$ has a local min at $(0,0)$?
Solution: $a > 0$ and $b^2 - 4ac < 0$.

(c) What conditions will ensure that $f(u,v)$ has a local max at $(0,0)$?
Solution: $a < 0$ and $b^2 - 4ac < 0$.

(d) What conditions will ensure that $f(u,v)$ has a saddle at $(0,0)$?
Solution: $b^2 - 4ac > 0$.

Second Derivative Test

If $f(x,y)$ is a function with a critical point at (a,b) then let

$$D = f_{xx}f_{yy}(a,b) - (f_{xy}(a,b))^2$$

- If $D > 0$ then f has either a local max or min at (a,b).
  - If $f_{xx}(a,b) > 0$ or $f_{yy}(a,b) > 0$ then f has a local min at (a,b).
  - If $f_{xx}(a,b) < 0$ or $f_{yy}(a,b) < 0$ then f has a local max at (a,b).

- If $D < 0$ then f has a saddle at (a, b).
- If $D = 0$ then you have no idea what is going on with f at (a, b).

Note 1: D is the negative of the discriminant in the textbook.

Note 2: D is the determinant of the Hessian, which you can read about on page 450 of the textbook.

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6. Use the second derivative test to analyze the functions at the critical points.

   (a) $f(x, y) = x^2 + 4y^2 - 4x$ at $(2, 0)$.
   Solution: $D = 16$. Local min at $(2, 0)$.

   (b) $f(x, y) = xy$ at $(0, 0)$.
   Solution: $D = -1$, saddle.

   (c) $f(x, y) = xy^2 - 6x^2 - 3y^2$ at $(0, 0)$ and $(3, -6)$.
   Solution: At $(0, 0)$, $D = 72$, local max.
   At $(3, -6)$, $D = -144$, saddle.

   (d) $f(x, y) = x^3 + y^3 - 6xy$ at $(2, 2)$ and $(0, 0)$.
   Solution: At $(2, 2)$, $D = 108$, local min.
   At $(0, 0)$, $D = -36$, saddle.