Math 233 - October 6, 2009

- Understand the nature of critical points for quadratic functions.
- Understand the second derivative test for functions of two variables.

1. Let $f(x, y) = 3x^3 + y^2 - 9x + 4y$.
   (a) Find all critical points for $f(x, y)$.
   (b) Find the second degree Taylor polynomial at each of the critical points
   (c) Determine the nature of the two critical points.

1. Let $f(x, y) = 3x^3 + y^2 - 9x + 4y$.
   (a) Find all critical points for $f(x, y)$.
   Solution: $(1, -2)$ and $(-1, -2)$
   (b) Find the second degree Taylor polynomial at each of the critical points
   Solution: At $(1, -2)$:

$$T_2(x, y) = -10 + \frac{1}{2}\left(18(x-1)^2 + 2(y+2)^2\right)$$

   At $(-1, -2)$:

$$T_2(x, y) = 2 + \frac{1}{2}\left(-18(x+1)^2 + 2(y+2)^2\right)$$

   (c) Determine the nature of the two critical points.
   Solution: Local min at $(1, -2)$, saddle at $(-1, -2)$.

Lecture Problems

2. Each of the functions below have a critical point at $(0,0)$ (why?). Determine if there is a local max, local min or saddle at $(0,0)$.

   (a) $f(x,y) = 4x^2 + 15y^2$.
   (b) $f(x,y) = 2x^2 + 91y^2$.
   (c) $f(x,y) = -2x^2 - 91y^2$.
   (d) $f(x,y) = -12x^2 - 5y^2$.
   (e) $f(x,y) = x^2 - y^2$.
   (f) $f(x,y) = -x^2 + y^2$.

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2. Each of the functions below have a critical point at $(0,0)$ (why?). Determine if there is a local max, local min or saddle at $(0,0)$.

   (a) $f(x,y) = 4x^2 + 15y^2$.
   Solution: Min at $(0,0)$.

   (b) $f(x,y) = 2x^2 + 91y^2$.
   Solution: Min at $(0,0)$.

   (c) $f(x,y) = -2x^2 - 91y^2$.
   Solution: Max at $(0,0)$.

   (d) $f(x,y) = -12x^2 - 5y^2$.
   Solution: Max at $(0,0)$.

   (e) $f(x,y) = x^2 - y^2$.
   Solution: Saddle at $(0,0)$.

   (f) $f(x,y) = -x^2 + y^2$.
   Solution: Saddle at $(0,0)$.

3. Starting with the function $f(x, y) = ax^2 + bxy + cy^2$ make the substitution $x = u - \frac{b}{2a}v$ and $y = v$ to transform the functions to functions of u and v. (Simplify your result.)

   (a) $f(x, y) = x^2 + xy + y^2$, $f(u, v) =$.
   (b) $f(x, y) = 2x^2 + xy + y^2$, $f(u, v) =$.
   (c) $f(x, y) = 3x^2 + xy + y^2$, $f(u, v) =$.
   (d) $f(x, y) = x^2 + 2xy + y^2$, $f(u, v) =$.
   (e) $f(x, y) = x^2 + 3xy + y^2$, $f(u, v) =$.
   (f) $f(x, y) = x^2 + xy - 2y^2$, $f(u, v) =$.
   (g) $f(x, y) = -3x^2 + xy - 5y^2$, $f(u, v) =$.

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   (a) $f(x, y) = x^2 + xy + y^2$, $f(u, v) = u^2 + \frac{3}{4}v^2$.
   (b) $f(x, y) = 2x^2 + xy + y^2$, $f(u, v) = 2u^2 + \frac{7}{8}v^2$.
   (c) $f(x, y) = 3x^2 + xy + y^2$, $f(u, v) = 3u^2 + \frac{11}{12}v^2$.
   (d) $f(x, y) = x^2 + 2xy + y^2$, $f(u, v) = u^2$.
   (e) $f(x, y) = x^2 + 3xy + y^2$, $f(u, v) = u^2 - \frac{5}{4}v^2$.
   (f) $f(x, y) = x^2 + xy - 2y^2$, $f(u, v) = u^2 - \frac{9}{4}v^2$.
   (g) $f(x, y) = -3x^2 + xy - 5y^2$, $f(u, v) = -3u^2 - \frac{39}{12}v^2$.

4. Determine if the functions in the previous problem have a local max, local min or saddle at $(0, 0)$.

   (a) $f(x, y) = x^2 + xy + y^2$,
   (b) $f(x, y) = 2x^2 + xy + y^2$,
   (c) $f(x, y) = 3x^2 + xy + y^2$,
   (d) $f(x, y) = x^2 + 2xy + y^2$,
   (e) $f(x, y) = x^2 + 3xy + y^2$,
   (f) $f(x, y) = x^2 + xy - 2y^2$,
   (g) $f(x, y) = -3x^2 + xy - 5y^2$,

4. Determine if the functions in the previous problem have a local max, local min or saddle at $(0, 0)$.

   (a) $f(x, y) = x^2 + xy + y^2$, **Solution:** Local Min
   (b) $f(x, y) = 2x^2 + xy + y^2$, **Solution:** Local Min
   (c) $f(x, y) = 3x^2 + xy + y^2$, **Solution:** Local Min
   (d) $f(x, y) = x^2 + 2xy + y^2$, **Solution:** Local Min
   (e) $f(x, y) = x^2 + 3xy + y^2$, **Solution:** Saddle
   (f) $f(x, y) = x^2 + xy - 2y^2$, **Solution:** Saddle
   (g) $f(x, y) = -3x^2 + xy - 5y^2$, **Solution:** Local Max

5. Let $f(x, y) = ax^2 + bxy + cy^2$. Assume $a \neq 0$.
   (a) Let $x = u - \frac{b}{2a}v$ and $y = v$.

   $$f(u, v) =$$

   (b) What conditions will ensure that $f(u, v)$ has a local min at $(0, 0)$?
   (c) What conditions will ensure that $f(u, v)$ has a local max at $(0, 0)$?
   (d) What conditions will ensure that $f(u, v)$ has a saddle at $(0, 0)$?

5. Let $f(x,y) = ax^2 + bxy + cy^2$. Assume $a \neq 0$.

   (a) Let $x = u - \frac{b}{2a}v$ and $y = v$.

$$f(u,v) = au^2 - \frac{(b^2 - 4ac)}{4a}v^2$$

   (b) What conditions will ensure that $f(u,v)$ has a local min at $(0,0)$?
   Solution: $a > 0$ and $b^2 - 4ac < 0$.

   (c) What conditions will ensure that $f(u,v)$ has a local max at $(0,0)$?
   Solution: $a < 0$ and $b^2 - 4ac < 0$.

   (d) What conditions will ensure that $f(u,v)$ has a saddle at $(0,0)$?
   Solution: $b^2 - 4ac > 0$.

Second Derivative Test

If $f(x, y)$ is a function with a critical point at (a, b) then let

$$D = f_{xx}f_{yy}(a, b) - (f_{xy}(a, b))^2$$

- If $D > 0$ then f has either a local max or min at (a, b).
  - If $f_{xx}(a, b) > 0$ or $f_{yy}(a, b) > 0$ then f has a local min at (a, b).
  - If $f_{xx}(a, b) < 0$ or $f_{yy}(a, b) < 0$ then f has a local max at (a, b).
- If $D < 0$ then f has a saddle at (a, b).
- If $D = 0$ then you have no idea what is going on with f at (a, b).

Note 1: D is the negative of the discriminant in the textbook.

Note 2: D is the determinant of the Hessian, which you can read about on page 450 of the textbook.

6. Use the second derivative test to analyze the functions at the critical points.

   (a) $f(x, y) = x^2 + 4y^2 - 4x$ at $(2, 0)$.
   (b) $f(x, y) = xy$ at $(0, 0)$.
   (c) $f(x, y) = xy^2 - 6x^2 - 3y^2$ at $(0, 0)$ and $(3, -6)$.
   (d) $f(x, y) = x^3 + y^3 - 6xy$ at $(2, 2)$ and $(0, 0)$.

6. Use the second derivative test to analyze the functions at the critical points.

   (a) $f(x, y) = x^2 + 4y^2 - 4x$ at $(2, 0)$.
   Solution: $D = 16$. Local min at $(2, 0)$.

   (b) $f(x, y) = xy$ at $(0, 0)$.
   Solution: $D = -1$, saddle.

   (c) $f(x, y) = xy^2 - 6x^2 - 3y^2$ at $(0, 0)$ and $(3, -6)$.
   Solution: At $(0, 0)$, $D = 72$, local max.
   At $(3, -6)$, $D = -144$, saddle.

   (d) $f(x, y) = x^3 + y^3 - 6xy$ at $(2, 2)$ and $(0, 0)$.
   Solution: At $(2, 2)$, $D = 108$, local min.
   At $(0, 0)$, $D = -36$, saddle.