Math 233 - October 5, 2009

Solutions

- How can we tell the difference between critical points? i.e., is it a max, min or saddle?
- Why does the second derivative test work in 1-variable?
- What can we learn from the second derivative test in 1-variable?

1. $f(x) = 3x^4 - 4x^3 + 1$.

   (a) Find all critical points of f. Use the second derivative test to determine if these are maxima or minima or neither.

P	$f''(P)$	Conclusion
$x = 0$	12	Min
$x = 1$	0	Inconclusive

   (b) For each critical point, find the second order Taylor polynomial at that point.

   At $x = 0$: $\quad T_2(x) = 1 + 6x^2$

   At $x = 1$: $\quad T_2(x) = 2$

   (c) Determine the nature of the critical point of the Taylor polynomials.

   Solution: At $x = 0$, the polynomial $1 + 6x^2$ has a minimum, just like the original function. At $x = 1$, the polynomial 2 is constant.

2. $f(x) = x^4 - 8x^2$.

   (a) Find all critical points of f. Use the second derivative test to determine if these are maxima or minima or neither.

P	$f''(P)$	Conclusion
$x = -2$	96	Min
$x = 0$	-48	Max
$x = 2$	96	Min

   (b) For each critical point, find the second order Taylor polynomial at that point.

   At $x = -2$: $\quad T_2(x) = -16 + 16(x + 2)^2$

   At $x = 0$: $\quad T_2(x) = -8x^2$

   At $x = 2$: $\quad T_2(x) = -16 + 16 * (x - 2)^2$

(c) Determine the nature of the critical point of the Taylor polynomials.

Solution: At $x = -2$, the polynomial $-16 + 16(x + 2)^2$ has a minimum, just like the original function.

At $x = 0$, the polynomial $-8x^2$ has a maximum, just like the original function.

At $x = 2$, the polynomial $-16 + 16(x - 2)^2$ has a minimum, just like the original function.

3. If $f(x) = a + bx^2$ (for $b \neq 0$).

   (a) What are the critical points of $f(x)$? **Solution:** Only one critical point at $x = 0$.

   (b) What values of b will make sure that you have a minimum at your critical point? **Solution:** $b > 0$

   (c) What values of b will make sure that you have a maximum at your critical point? **Solution:** $b < 0$

Lecture Problems

4. Let $P = (a, b)$ and suppose you are given a function $f(x, y)$. What is the general formula for the first order Taylor series for the function at the point P?

$$T_1(x, y) = f(P) + f_x(P)(x - a) + f_y(P)(y - b)$$

5. Find the first order Taylor series for the functions at the given points

   (a) $f(x, y) = y^3 + x^2y - 1$ at $(1, -2)$.

$$T_1(x, y) = 1 - 4(x - 1) + 13(y + 2)$$

   (b) $f(x, y) = \ln(1 - x + y)$ at $(1, 3)$.

$$T_1(x, y) = \ln 3 - \frac{1}{3}(x - 1) + \frac{1}{3}(y - 3)$$

6. Find the second order Taylor series for the functions at the given points

   (a) $f(x, y) = y^3 + x^2y - 1$ at $(1, -2)$.

$$T_2(x, y) = T_1(x, y) + \frac{1}{2}\left(-4(x - 1)^2 + 2(2)(x - 1)(y + 2) - 12(y + 2)^2\right)$$

   (b) $f(x, y) = \ln(1 - x + y)$ at $(1, 3)$.

$$T_2(x, y) = T_1(x, y) + \frac{1}{2}\left(-\frac{1}{9}(x - 1)^2 + \frac{2}{9}(x - 1)(y - 3) - \frac{1}{9}(y - 3)^2\right)$$