Math 233 Warm Up Problems

October 2, 2009

Solutions

1. Find the extrema of $f(x, y, z) = x^2 + y^2 + z^2$ subject to $x + 3y - 2z = 12$.

   (a) Set up the equations required by the method of Lagrange multipliers.

   $$\begin{aligned} 2x &= \lambda \\ 2y &= 3\lambda \\ 2z &= -2\lambda \\ x + 3y - 2z &= 12 \end{aligned}$$

   (b) Solve the equations
   Solution: The only point is:

   $$\left(\frac{6}{7}, \frac{18}{7}, \frac{-12}{7}\right)$$

   (c) Analyze the critical points
   Solution: The point given is at a minimum, there is no maximum of the function.

2. Using Lagrange multipliers, find the extrema of $f(x, y, z) = 4x - 2y + 3z$ subject to $2x^2 + y^2 - 3z = 0$.

   Solution: Start with the equations

   $$\begin{aligned} 4 &= 4x\lambda \\ -2 &= 2y\lambda \\ 3 &= -3\lambda \\ -3z + y^2 + 2x^2 &= 0 \end{aligned}$$

   and solve to get the critical point $(-1, 1, 1)$, which will be a minimum (why?). There is no maximum.

3. Using Lagrange multipliers, find the minimal distance from the origin to the plane $x+3y-2z = 4$.

   Solution: Find the minimum of $r = \sqrt{x^2 + y^2 + z^2}$ subject to $x + 3y - 2z = 4$.

   But, we will actually find the minimum of $f(x, y, z) = x^2 + y^2 + z^2$ subject to $x + 3y - 2z = 4$.

   The minimum of $\sqrt{8/7}$ occurs at

   $$\left(\frac{2}{7}, \frac{6}{7}, \frac{-4}{7}\right)$$

4. Find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $x + y + z = 1$. Assume that $x, y, z \geq 0$.

   Solution: Set up the equations

$$
\begin{aligned}
yz =& \lambda \\
xz =& \lambda \\
xy =& \lambda \\
x + y + z =& 1
\end{aligned}
$$

   Solve the equations, making sure we only take positive x, y, z which gives solutions

P	$f(P)$	Conclusion
$(0, 1, 0)$	0	Min
$(1, 0, 0)$	0	Min
$(0, 0, 1)$	0	Min
$(1/3, 1/3, 1/3)$	$1/27$	Max

5. A rectangular box whose edges are parallel to the coordinate axes is inscribed in the ellipsoid $36x^2 + 4y^2 + 9z^2 = 36$. Find the maximum possible volume for such a box.

   Solution: We want to maximize volume, which is $V = 8xyz$, subjec to $36x^2 + 4y^2 + 9z^2 = 36$. This leads to equations

$$
\begin{aligned}
8yz =& 72\lambda x \\
8xz =& 8\lambda y \\
8xy =& 18\lambda z \\
36x^2 + 4y^2 + 9z^2 =& 36
\end{aligned}
$$

   There are several solutions with $x, y, z = 0$. The solution with positive x, y, z is

$$\left(\frac{1}{\sqrt{3}}, \sqrt{3}, \frac{2}{\sqrt{3}}\right)$$

   and thus the maximal volume is $\frac{16}{\sqrt{3}}$.

6. Using Lagrange multipliers, find the minimal distance from the origin to the surface $x^2y - z^2 + 9 = 0$.

   Solution: Find the minimum of $f(x, y, z) = x^2 + y^2 + z^2$ subject to $x^2y - z^2 + 9 = 0$. This leads to equations

$$
\begin{aligned}
2x &= 2xy\lambda \\
2y &= x^2\lambda \\
2z &= -2z\lambda \\
-z^2 + x^2y + 9 &= 0
\end{aligned}
$$

This leads to the following

P	$f(P)$	Conclusion
$(0,0,3)$	9	
$(0,0,-3)$	9	
$(\sqrt{2},-1,\sqrt{7})$	10	
$(\sqrt{2},-1,-\sqrt{7})$	10	
$(-\sqrt{2},-1,\sqrt{7})$	10	
$(-\sqrt{2},-1,-\sqrt{7})$	10	
$(-(36^{1/3},-(9/2)^{1/3},0)$	≈ 8.17	Min
$((36^{1/3},-(9/2)^{1/3},0)$	≈ 8.17	

Lecture Problems

7. Find the max/min of $f(x,y,z) = 4y - 2z$ subject to $2x - y - z = 2$ and $x^2 + y^2 = 1$.

   Solution: Set up the equations

$$\begin{aligned} 0 &= 2\lambda + 2\mu x \\ 4 &= -\lambda + 2\mu y \\ -2 &= -\lambda \\ 2x - y - z &= 2 \\ x^2 + y^2 &= 1 \end{aligned}$$

   Solve to find solutions

P	$f(P)$	Conclusion
$\left(\frac{-2}{\sqrt{13}}, \frac{3}{\sqrt{13}}, -2 - \frac{7}{\sqrt{13}}\right)$	$4 + \frac{26}{\sqrt{13}}$	Max
$\left(\frac{2}{\sqrt{13}}, \frac{-3}{\sqrt{13}}, -2 + \frac{7}{\sqrt{13}}\right)$	$4 - \frac{26}{\sqrt{13}}$	Min

8. Find the maximum and minimum of $f(x,y,z) = x + 2y + 3z$ on the intersection of the cylinder $x^2 + y^2 = 2$ and the plane $y + z = 1$.

   Solution: Max of 5 at $(1,-1,2)$. Min of 1 at $(-1,1,0)$.