Math 233 Warm Up Problems
October 2, 2009
Solutions

1. Find the extrema of \( f(x, y, z) = x^2 + y^2 + z^2 \) subject to \( x + 3y - 2z = 12 \).

   (a) Set up the equations required by the method of Lagrange multipliers.

   \[
   \begin{align*}
   2x &= \lambda \\
   2y &= 3\lambda \\
   2z &= -2\lambda \\
   x + 3y - 2z &= 12
   \end{align*}
   \]

   (b) Solve the equations

   **Solution:** The only point is:

   \[
   \left( \frac{6}{7}, \frac{18}{7}, \frac{-12}{7} \right)
   \]

   (c) Analyze the critical points

   **Solution:** The point given is at a minimum, there is no maximum of the function.

2. Using Lagrange multipliers, find the extrema of \( f(x, y, z) = 4x - 2y + 3z \) subject to \( 2x^2 + y^2 - 3z = 0 \).

   **Solution:** Start with the equations

   \[
   \begin{align*}
   4 &= 4x\lambda \\
   -2 &= 2y\lambda \\
   3 &= -3\lambda \\
   -3z + y^2 + 2x^2 &= 0
   \end{align*}
   \]

   and solve to get the critical point \((-1, 1, 1)\), which will be a minimum (why?). There is no maximum.

3. Using Lagrange multipliers, find the minimal distance from the origin to the plane \( x + 3y - 2z = 4 \).

   **Solution:** Find the minimum of \( r = \sqrt{x^2 + y^2 + z^2} \) subject to \( x + 3y - 2z = 4 \).

   But, we will actually find the minimum of \( f(x, y, z) = x^2 + y^2 + z^2 \) subject to \( x + 3y - 2z = 4 \).

   The minimum of \( \sqrt{8/7} \) occurs at

   \[
   \left( \frac{2}{7}, \frac{6}{7}, \frac{-4}{7} \right)
   \]
4. Find the maximum and minimum values of \( f(x, y, z) \) subject to the constraint \( x + y + z = 1 \). Assume that \( x, y, z \geq 0 \).

**Solution:** Set up the equations

\[

cpy = yz \\

cpy = xz \\

cpy = xy \\
\]

\[
x + y + z = 1
\]

Solve the equations, making sure we only take positive \( x, y, z \) which gives solutions

<table>
<thead>
<tr>
<th>( P )</th>
<th>( f(P) )</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 1, 0)</td>
<td>0</td>
<td>Min</td>
</tr>
<tr>
<td>(1, 0, 0)</td>
<td>0</td>
<td>Min</td>
</tr>
<tr>
<td>(0, 0, 1)</td>
<td>0</td>
<td>Min</td>
</tr>
<tr>
<td>(1/3, 1/3, 1/3)</td>
<td>1/27</td>
<td>Max</td>
</tr>
</tbody>
</table>

5. A rectangular box whose edges are parallel to the coordinate axes is inscribed in the ellipsoid \( 36x^2 + 4y^2 + 9z^2 = 36 \). Find the maximum possible volume for such a box.

**Solution:** We want to maximize volume, which is \( V = 8xyz \), subject to \( 36x^2 + 4y^2 + 9z^2 = 36 \). This leads to equations

\[
8yz = 72\lambda x \\
8xz = 8\lambda y \\
8xy = 18\lambda z \\
36x^2 + 4y^2 + 9z^2 = 36
\]

There are several solutions with \( x, y, z = 0 \). The solution with positive \( x, y, z \) is

\[
\left( \frac{1}{\sqrt{3}}, \sqrt{3}, \frac{2}{\sqrt{3}} \right)
\]

and thus the maximal volume is \( \frac{16}{\sqrt{3}} \).

6. Using Lagrange multipliers, find the minimal distance from the origin to the surface \( x^2y - z^2 + 9 = 0 \).

**Solution:** Find the minimum of \( f(x, y, z) = x^2 + y^2 + z^2 \) subject to \( x^2y - z^2 + 9 = 0 \). This leads to equations

\[
2x = 2xy\lambda \\
2y = x^2\lambda \\
2z = -2z\lambda \\
-z^2 + x^2y + 9 = 0
\]
This leads to the following

<table>
<thead>
<tr>
<th>$P$</th>
<th>$f(P)$</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0, 0, 3)$</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>$(0, 0, -3)$</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>$(\sqrt{2}, -1, \sqrt{7})$</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>$(\sqrt{2}, -1, -\sqrt{7})$</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>$(-\sqrt{2}, -1, \sqrt{7})$</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>$(-\sqrt{2}, -1, -\sqrt{7})$</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>$(-36^{1/3}, -(9/2)^{1/3}, 0)$</td>
<td>$\approx 8.17$</td>
<td>Min</td>
</tr>
<tr>
<td>$((36^{1/3}, -(9/2)^{1/3}, 0)$</td>
<td>$\approx 8.17$</td>
<td></td>
</tr>
</tbody>
</table>

Lecture Problems

7. Find the max/min of $f(x, y, z) = 4y - 2z$ subject to $2x - y - z = 2$ and $x^2 + y^2 = 1$.

Solution: Set up the equations

\[
\begin{align*}
0 &= 2\lambda + 2\mu x \\
4 &= -\lambda + 2\mu y \\
-2 &= -\lambda \\
2x - y - z &= 2 \\
x^2 + y^2 &= 1
\end{align*}
\]

Solve to find solutions

<table>
<thead>
<tr>
<th>$P$</th>
<th>$f(P)$</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\left( -\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}, -2 - \frac{7}{\sqrt{13}} \right)$</td>
<td>$4 + \frac{26}{\sqrt{13}}$</td>
<td>Max</td>
</tr>
<tr>
<td>$\left( \frac{2}{\sqrt{13}}, -\frac{3}{\sqrt{13}}, -2 + \frac{7}{\sqrt{13}} \right)$</td>
<td>$4 - \frac{26}{\sqrt{13}}$</td>
<td>Min</td>
</tr>
</tbody>
</table>

8. Find the maximum and minimum of $f(x, y, z) = x + 2y + 3z$ on the intersection of the cylinder $x^2 + y^2 = 2$ and the plane $y + z = 1$.

Solution: Max of 5 at $(1, -1, 2)$. Min of 1 at $(-1, 1, 0)$. 