Math 233 - November 13, 2009
Solutions

- More versions of Green's Theorem

$$\oint_{\partial R} p\, dx + q\, dy = \iint_R \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\, dA$$
$$= \iint_R \text{Rot }(p,q)\, dA$$
$$= \int_{\partial R} F \cdot \mathbf{u}\, ds$$
$$\oint_{\partial R} F \cdot \mathbf{n}\, ds = \iint_R \text{Div } F\, dA$$

1. Let $r(t) = (t, t^3)$.

   (a) Find a tangent vector to the curve for any time t.
   Solution: $r'(t) = (1, 3t^2)$

   (b) Find a vector normal to the curve for any time t.
   Solution: There are two possibilities

$$N_1 = (3t^2, -1) \qquad \text{Right normal vector}$$
$$N_2 = (-3t^2, 1) \qquad \text{Left normal vector}$$

   (c) Does your normal vector point to the right or to the left of the direction of the curve. Find the normal vector in the other direction too.

   (d) Find both unit normal vectors **Solution:** There are two possibilities

$$n_1 = \frac{1}{1+9y^4}(3t^2, -1) \qquad \text{Right unit normal vector}$$
$$n_2 = \frac{1}{1+9y^4}(-3t^2, 1) \qquad \text{Right unit normal vector}$$

2. A wire in the shape of a semi-circle, $r(t) = (3\cos t, 3\sin t)$, $0 \le t \le \pi$. has density equal to $\delta(x,y) = y$ (in other words, the wire is heavier as y increases). Using an integral $\int_C \delta(x,y)\, ds$, find the mass of the wire.

$$M = \int_C y\, ds = \int_0^\pi 3\sin t\sqrt{9\sin^2 t + 9\cos^2 t}\, dt = 18$$

Lecture Problems

3. Let $F = (x^2 + y^2, xy)$ and C be the boundary of the square with vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$. Using Green's Theorem, calculate the circulation of F around C.

$$\oint_C F \cdot \mathbf{u}\, ds = \iint_R \text{Rot } F\, dA = \int_0^1 \int_0^1 -y\, dA = -\frac{1}{2}$$

4. Let $F = (-y, x)$ and let r be the circle of radius 2 center at the origin. Find the flux of F across r.

$$\int_r F \cdot n\, ds = \int_0^{2\pi} (-2\sin t, 2\cos t) \cdot \frac{(2\cos t, 2\sin t)}{2}\, 2\, dt = 0$$

5. Let $F = (2y, xy)$. Find the flux across the unit circle.

$$\int_r F \cdot n\, ds = \int_0^{2\pi} (2\cos t, \cos t \sin t) \cdot (\cos t, \sin t)\, dt = 2\pi$$

6. Use Green's Theorem to comute the flux of the vector field $F = (2x, xy)$ across the unit circle.

$$\int_r F \cdot n\, ds = \iint_R 2 + x\, dA = \int_0^{2\pi} \int_0^1 (2 + r\cos\theta) r\, dr\, d\theta = 2\pi$$