

Math 233 - November 10, 2009

Solutions

- Parametrizing curves
- Proof of Green's Theorem
- General version of Green's Theorem

1. Parametrize the following curves

- (a) The curve that is the graph of $y = x^3$ between $x = 1$ and $x = 4$, parametrized from left to right.

$$r(t) = (t, t^3) \quad 1 \leq t \leq 4$$

- (b) The curve that is the graph of $x = \sin y$ between $y = 0$ and $y = \pi$, parametrized from bottom to top.

$$r(t) = (\sin t, t) \quad 0 \leq t \leq \pi$$

- (c) The curve that is the graph of the polar equation $r = 1 + \sin \theta$.

$$r(\theta) = ((1 + \sin \theta) \cos \theta, (1 + \sin \theta) \sin \theta) \quad 0 \leq \theta \leq 2\pi$$

2. Find the area enclosed by the graph $r = 1 + \sin \theta$.

Can you do this by setting up a line integral?

Solution: Standard way:

$$A = \int_0^{2\pi} \int_0^{1+\sin \theta} r \, dr \, d\theta = \int_0^{2\pi} \frac{1}{2} (1 + \sin \theta)^2 \, d\theta = \frac{3\pi}{2}$$

Line integral, use the parametrization from Problem 1:

$$\begin{aligned} A &= \frac{1}{2} \oint_r -y \, dx + x \, dy \\ &= \frac{1}{2} \int_0^{2\pi} -(1 + \sin \theta) \sin \theta (\cos^2 t - \sin^2 t - \sin t) \\ &\quad + (1 + \sin \theta) \cos \theta (2 \cos t \sin t + \cos t) \, dt \\ &= \frac{3\pi}{2} \end{aligned}$$

Lecture Problems

3. Let C be the rectangle with vertices $(2, 1)$, $(6, 1)$, $(6, 4)$ and $(2, 4)$, oriented positively. Use Green's Theorem to find the line integral:

$$\oint_C (e^{3x} + 2y) \, dx + (x^2 + \sin y) \, dy = \int_2^6 \int_1^4 2x - 2 \, dy \, dx = 72$$

4. Let C be the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$. Find the work done by the vector field $F = (y, x^2y)$ around C in the counter-clockwise direction.

Solution:

$$\oint_C F \cdot dr = \int_0^1 \int_0^1 2xy - 1 \, dA = -\frac{1}{2}$$