Name:
ID:
Section:

This exam has 20 multiple choice questions worth 5 points each.
Important:

- No graphing calculators! Any non-graphing scientific calculator is fine.
- For the multiple choice questions, mark your answer on the answer card.
- Show all your work for the written problems. You will be graded on the ease of reading your solution as well as for your work.
- You are allowed both sides of 3×5 note "cheat card" for the exam.

1. Find the equation of the plane through the point $(1, 2, 4)$ and parallel to the plane $2x+y-3z = 18$.

   (a) $2x + y - 3z = -18$
   (b) $2x + y - 3z = -8$ $\longrightarrow$ **CORRECT**
   (c) $2x + y - 3z = 8$
   (d) $2x + y - 3z = 18$
   (e) $x + 2y + 4z = -21$
   (f) $x + 2y + 4z = -18$
   (g) $x + 2y + 4z = 18$
   (h) $x + 2y + 4z = 21$

   Solution: Parallel planes have parallel normal vectors. Thus the plane must have equation $x + y - 3z = D$. Plug in the point $(1, 2, 4)$ to find $D = -8$.

2. Find the vector projection of the vector $\mathbf{a} = (3, 0, 3)$ onto the vector $\mathbf{b} = (1, -1, 1)$.

   (a) $\left(\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}\right)$

   (b) $(1, -1, 1)$

   (c) $(2, -2, 2)$ $\longrightarrow$ **CORRECT**

   (d) $(1, 0, 1)$

   (e) $(3, 0, 3)$

   (f) $(6, 0, 6)$

   (g) $(4, -1, 4)$

   Solution:

$$\mathrm{Pr}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}}\, \mathbf{b} = 2(1, -1, 1) = (2, -2, 2)$$

3. Find the minimal distance from the origin to the plane $x + 5y - z = 6$.

   (a) 0

   (b) $\dfrac{2}{\sqrt{5}}$

   (c) $\dfrac{2}{5}$

   (d) $\dfrac{2}{\sqrt{3}}$ $\longrightarrow$ **CORRECT**

   (e) $\dfrac{2}{3}$

   (f) $\dfrac{2}{\sqrt{2}}$

   (g) 1

   Solution: This can be solved using Lagrange multiplier techniques, but it can also be solved using vector techniques (see page 42 of the text).

$$d = \frac{(6,0,0)\cdot(1,5,-1)}{\|(1,5,-1)\|} = \frac{1}{\sqrt{3}}$$

4. Which of the following sets in $\mathbb{R}$ are closed?

   I. $[0, 3]$
   II. $\mathbb{R}$
   III. $[-1, 1)$
   IV. $\mathbb{Q}$, the set of all rational numbers (fractions)

   (a) I only
   (b) II only
   (c) III only
   (d) IV only
   (e) I and II only $\longrightarrow$ **CORRECT**
   (f) I and III only
   (g) I and IV only
   (h) II and III only
   (i) III and IV only
   (j) Exactly three are closed
   (k) Some other combination

   Solution: The more difficult one here is the set of rational numbers. Every point of $\mathbb{R}$ is a boundary point for $\mathbb{Q}$ and therefore $\mathbb{Q}$ does not contain all of its boundary points.

5. Which best describes the spherical equation $\rho = 8\cos\theta\sin\phi$?

   (a) A sphere centered at the origin

   (b) A sphere, not centered at the origin $\longrightarrow$ **CORRECT**

   (c) A cone

   (d) A plane

   (e) A paraboloid

   (f) A saddle

   (g) None of the above

   Solution: This is the equation $x^2 + y^2 + z^2 = 8x$, which is a sphere centered at the point $(4, 0, 0)$.

6. Find where the tangent plane to the surface $x^2 - y^3 + z^4 = 4$ at the point $(2, 1, 1)$ hits the y-axis.

   (a) -3 $\longrightarrow$ **CORRECT**
   (b) -2
   (c) -1
   (d) 0
   (e) 1
   (f) 2
   (g) 3
   (h) 4
   (i) 5
   (j) 6

   Solution: The tangent plane is $4z - 3y + 4x = 9$ which hits the y-axis at $y = -3$.

7. Let $f(x, y) = x^3 y$. Use a linearization at $(1, 2)$ to approximate $f(1.2, 1.3)$.

   (a) 2.0
   (b) 2.1
   (c) 2.2
   (d) 2.3
   (e) 2.4
   (f) 2.5 $\longrightarrow$ **CORRECT**
   (g) 2.6
   (h) 2.7
   (i) 2.8
   (j) 2.9

   Solution: $L(x, y) = 2 + 6(x - 1) + (y - 2)$, $L(1, 2, 1.3) = 2.5$

8. Find the directional derivative of $f(x,y) = x^3y - xy^2$ at the point $(1,3)$ in the direction of the vector $(-3,4)$.

   (a) -8
   (b) -7
   (c) -6
   (d) -5
   (e) -4 $\longrightarrow$ **CORRECT**
   (f) -3
   (g) -2
   (h) -1
   (i) 0
   (j) 1

   Solution: Remember to scale the vector to a unit vector.

$$Df_u(p) = \nabla f(p) \cdot u = (0,-5) \cdot \frac{1}{5}(-3,4) = -4$$

9. Find the maximum of $f(x, y) = x^2 + x + y^2$ on the disk $x^2 + y^2 \leq 1$.

   (a) 3
   (b) -2
   (c) -1
   (d) $-\dfrac{1}{2}$
   (e) $-\dfrac{1}{4}$
   (f) 0
   (g) $\dfrac{1}{4}$
   (h) $\dfrac{1}{2}$
   (i) 1
   (j) 2 $\longrightarrow$ **CORRECT**
   (k) 3

   Solution: The only critical point on the interior of the disk is a local minimum. The boundary of the disk can be parametrized as $r(t) = (\cos t, \sin t)$. $f(r(t)) = 1 + \cos t$, which has a maximum of 2.

10. The function $f(x,y) = x^4 + y^4 - 16xy - 2$ has 3 critical points: $(-2,-2)$, $(0,0)$, and $(2,2)$. These critical points are

(a) 3 local maximums

(b) 2 local maximums, 1 local minimum

(c) 2 local maximums, 1 saddle

(d) 1 local maximum, 2 local minimums

(e) 1 local maximum, 1 local minimum, 1 saddle

(f) 1 local maximum, 2 saddles

(g) 3 local minimums

(h) 2 local minimums, 1 saddle $\longrightarrow$ **CORRECT**

(i) 1 local minimum, 2 saddles

(j) 3 saddles

Solution:

p	$D = f_{xx}f_{yy} - f_{xy}^2$	$f_{xx}(p)$	Conclusion
$(2,2)$	2048	48	Min
$(-2,-2)$	2048	48	Min
$(0,0)$	-256	0	Saddle

11. Let C be the boundary of the triangle with vertices $(1,0)$, $(0,1)$ and $(-1,-1)$ oriented counter-clockwise. Let

$$G = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

Compute $\displaystyle\int_C G \cdot dr$

(a) -2π

(b) $-\pi$

(c) 0

(d) $\dfrac{\pi}{2}$

(e) π

(f) $\dfrac{3\pi}{2}$

(g) 2π $\longrightarrow$ **CORRECT**

(h) 3π

Solution: By Green's Theorem, this is equal to the line integral around the unit circle, which is equal to 2π.

12. Compute

$$\int_C (2x + 3y)\ dx + (3x - 8y)\ dy$$

where C is the union of the line segments from $(2, 3)$ to $(-2, 4)$ and then to $(1, 1)$.

(a) 6
(b) 7
(c) 8
(d) 9
(e) 10
(f) 11
(g) 12
(h) 13
(i) 14 $\longrightarrow$ **CORRECT**
(j) 15

Solution: A potential function for the vector field is $f(x, y) = x^2 + 3xy - 4y^2$.

$$\int_C (2x + 3y)\ dx + (3x - 8y)\ dy = f(1, 1) - f(2, 3) = 14$$

13. Let C be the curve with parametrization $r(t) = (t, t^2)$ for $0 \leq t \leq 2$. Compute $\int_C 3xy\ dx - 2x^2\ dy$

   (a) -9
   (b) -8
   (c) -7
   (d) -6
   (e) -5
   (f) -4 $\longrightarrow$ **CORRECT**
   (g) -3
   (h) -2
   (i) -1
   (j) 0

Solution:

$$\int_C 3xy\ dx - 2x^2\ dy = \int_0^2 -t^3\ dt = -4$$

14. Let C be the boundary of the square with vertices $(0,0)$, $(2,0)$, $(2,2)$, $(0,2)$ oriented counter-clockwise. Find $\int_C x^2\, dx + xy\, dy$.

    (a) 1

    (b) 2

    (c) 3

    (d) 4 $\longrightarrow$ **CORRECT**

    (e) 5

    (f) 6

    (g) 7

    (h) 8

    (i) 9

    (j) 10

    Solution: Use Green's Theorem

$$\int_C x^2\, dx + xy\, dy = \int_0^2 \int_0^2 y\, dy\, dx = 4$$

15. Let $f(x, y, z)$ be a smooth function. Let $C(t)$ be a smooth curve in $\mathbb{R}^3$, $0 \leq t \leq 1$, be completely contained in the level surface $f(x, y, z) = 5$. What is

$$\int_C \nabla f \cdot dr$$

(a) -2π

(b) -5

(c) $0 \longrightarrow$ **CORRECT**

(d) 5

(e) 2π

(f) Can not determine with the given information

Solution: Since f is the potential function, the value at the beginning and ending points of the curve are the same. Therefore the line integral is the difference between these, equal to 0.

16. Compute the average value of $f(x, y) = x + 2y$ over the box $[0, 2] \times [0, 2]$

    (a) 0
    (b) 1
    (c) 2
    (d) 3 $\longrightarrow$ **CORRECT**
    (e) 4
    (f) 6
    (g) 8
    (h) 10
    (i) 12

    Solution:

$$f_{ave} = \frac{1}{4}\int_0^2 \int_0^2 x + 2y \; dy \; dx = 3$$

17. A solid portion of the spherical shell lies in the first octant between $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 9$. The density of the solid has density function $\delta(x, y, z) = \dfrac{1}{x^2 + y^2 + z^2}$.

    Find the mass of the solid.

    (a) 0

    (b) $\dfrac{\pi}{4}$

    (c) $\dfrac{\pi}{2}$ $\longrightarrow$ **CORRECT**

    (d) π

    (e) 2π

    (f) 3π

    Solution:

$$M = \int_0^{\pi/2} \int_0^{\pi/2} \int_2^3 \frac{1}{\rho^2} \rho^2 \sin\phi \; d\rho \; d\phi \; d\theta = \frac{\pi}{2}$$

18. A tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(1,1,0)$, and $(0,1,1)$ has density $\delta(x,y,z) = 1$. Compute the mass of the tetrahedron.

    (a) $\dfrac{1}{12}$

    (b) $\dfrac{1}{10}$

    (c) $\dfrac{1}{9}$

    (d) $\dfrac{1}{8}$

    (e) $\dfrac{1}{6}$ $\longrightarrow$ **CORRECT**

    (f) $\dfrac{1}{5}$

    (g) $\dfrac{1}{4}$

    (h) $\dfrac{1}{3}$

    (i) $\dfrac{1}{2}$

    (j) 1

    Solution:

$$M = \int_0^1 \int_0^{1-x} \int_z^{x+z} 1 \; dy \; dz \; dx = \frac{1}{6}$$

19. Let R be the region in $\mathbb{R}^3$ bounded by the cylinder $x^2+y^2=9$, the planes $z=-1$ and $z=x+3$. Find $\iiint_R \sqrt{x^2+y^2}\, dV$.

   (a) 0
   (b) 9π
   (c) 18π
   (d) 27π
   (e) 36π
   (f) 45π
   (g) 54π
   (h) 63π
   (i) 72π $\longrightarrow$ **CORRECT**
   (j) 90π

Solution:

$$\int_0^{2\pi}\int_0^3\int_{-1}^{r\cos\theta+3} r\ dz\ dr\ d\theta = 72\pi$$

20. Which of the following integrals represent the volume of the portion of the unit sphere centered at the origin that lies in the first octant?

I. $\displaystyle\int_0^1 \int_0^{\sqrt{1-y^2}} \sqrt{1-x^2-y^2}\,dx\,dy$

II. $\displaystyle\int_0^{\frac{\pi}{2}} \int_0^1 r\sqrt{1-r^2}\,dr\,d\theta$

III. $\displaystyle\int_0^{\frac{\pi}{2}} \int_0^1 \int_0^{\sqrt{1-r^2}} r\,dz\,dr\,d\theta$

IV. $\displaystyle\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$

(a) Exactly one of these
(b) I and II only
(c) I and III only
(d) I and IV only
(e) I, II and III only
(f) I, II and IV only
(g) I, III and IV only
(h) II, III and IV only
(i) All $\longrightarrow$ **CORRECT**
(j) None