Name:
ID:
Section:

This exam has 17 questions:

- 15 multiple choice questions worth 5 points each.
- 2 hand graded questions worth 25 points total.

Important:

- No graphing calculators! Any non-graphing scientific calculator is fine.
- For the multiple choice questions, mark your answer on the answer card.
- Show all your work for the written problems. You will be graded on the ease of reading your solution as well as for your work.
- You are allowed both sides of 3×5 note "cheat card" for the exam.

1. Let $r(t)$ be the line from $(1, 2)$ to $(4, -2)$. Compute

$$\int_r y^2 \, dx - x \, dy$$

(a) Something less than 8

(b) 8

(c) 9

(d) 10

(e) 11

(f) 12

(g) 13

(h) 14 $\longrightarrow$ **CORRECT**

(i) 15

(j) Something greater than 15

(k) Unable to determine with the given information

Solution: Parametrize the line $r(t) = (1, 2) + t(3, -4)$.

$$\int_r y^2 \, dx - x \, dy = \int_0^1 (2 - 4t)^2(3) - (3t + 1)(-4) \, dt = 14$$

2. Compute the work done by the vector field $F = (y^2, x^2, 2z)$ along a curve that travels from the origin to $(1, 2, 5)$.

   (a) Something less than 25
   (b) 25
   (c) 26
   (d) 27
   (e) 28
   (f) 29
   (g) 30
   (h) 31
   (i) 32
   (j) Something greater than 32
   (k) Unable to determine with the given information $\longrightarrow$ **CORRECT**

   Solution: This is not a conservative field and is thus dependent on the path.

3. Compute the work done by the vector field $F = (y, x, 2)$ along a curve that travels from the origin to $(1, 2, 5)$.

   (a) Something less than 5
   (b) 5
   (c) 6
   (d) 7
   (e) 8
   (f) 9
   (g) 10
   (h) 11
   (i) 12 $\longrightarrow$ **CORRECT**
   (j) Something greater than 12
   (k) Unable to determine with the given information

Solution: This is a conservative field with potential function $\phi(x, y, z) = xy + 2z$. Thus the work is

$$\int_{(0,0,0)}^{(1,2,5)} y\ dx + x\ dy + 2\ dz = \phi(1, 2, 5) - \phi(0, 0, 0) = 12$$

4. Let C be the boundary of the rectangle $R = [1,2] \times [0,3]$ where C is parametrized counter-clockwise. Compute

$$\int_C (e^{x^2} - x^2 y)\, dx + (x + \sin y^3)\, dy$$

(a) Something less than 10

(b) 10 $\longrightarrow$ **CORRECT**

(c) 11

(d) 12

(e) 13

(f) 14

(g) 15

(h) 16

(i) 17

(j) 18

(k) Something greater than 18

Solution: This is easiest to compute using Green's Theorem

$$\int_C (e^{x^2} - x^2 y)\, dx + (x + \sin y^3)\, dy = \iint_R (1 + x^2)\, dA = 10$$

5. Let

$$F = \left(\frac{x-y}{x^2+y^2}, \frac{x+y}{x^2+y^2} \right)$$

Let $r(t)$ be the positively oriented (counter-clockwise) unit circle. Compute

$$\frac{1}{2\pi} \int_r F \cdot dr$$

(a) -3
(b) -2
(c) -1
(d) 0
(e) 1 $\longrightarrow$ **CORRECT**
(f) 2
(g) 3
(h) Integral is undefined

Solution: If $G = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$ then $F = G + \nabla\phi$ where $\nabla\phi = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right)$, which is a conservative field $\phi = \frac{1}{2}\ln(x^2+y^2)$.
Thus, by Theorem 4.3 on page 231,

$$k = \frac{1}{2\pi} \int_r F \cdot dr = 1$$

6. Let $C(t)$ be the curve traced out, in the counter-clockwise direction, along the triangle with vertices $(-4,-2)$, $(2,-2)$ and $(1,2)$. Compute

$$\int_C 3y\ dx + 4x\ dy$$

(a) 7
(b) 8
(c) 9
(d) 10
(e) 11
(f) 12 $\longrightarrow$ **CORRECT**
(g) 13
(h) 14
(i) 15
(j) 16
(k) 17

Solution: By Green's Theorem, this is just the area of the region.

$$\int_C -y\ dx + x\ dy = \iint_R (4-3)\ dA = \text{Area}(R) = \frac{1}{2}(6)(4) = 12$$

7. Suppose $F = (f, g)$ is a differentiable vector field defined on an open subset $U \subset \mathbb{R}^2$ with

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 0$$

Which of the following are true?

I. Let $C(t)$ be a closed path (a loop) in U. Then $\int_C F \cdot dr = 0$.

II. Let $C(t)$ be a path in U with C^- the reverse path of C. Then $\int_C F \cdot dr + \int_{C^-} F \cdot dr = 0$

III. Let $C(t)$ be a closed path (a loop) in U and let R be some rectangle completely contained in U. If C is completely contained in R then $\int_C F \cdot dr = 0$.

(a) I only

(b) II only

(c) III only

(d) I and II only

(e) I and III only

(f) II and III only $\longrightarrow$ **CORRECT**

(g) All are true

(h) None are true

Solution:

I. False. The "special vector field" is the counter example.

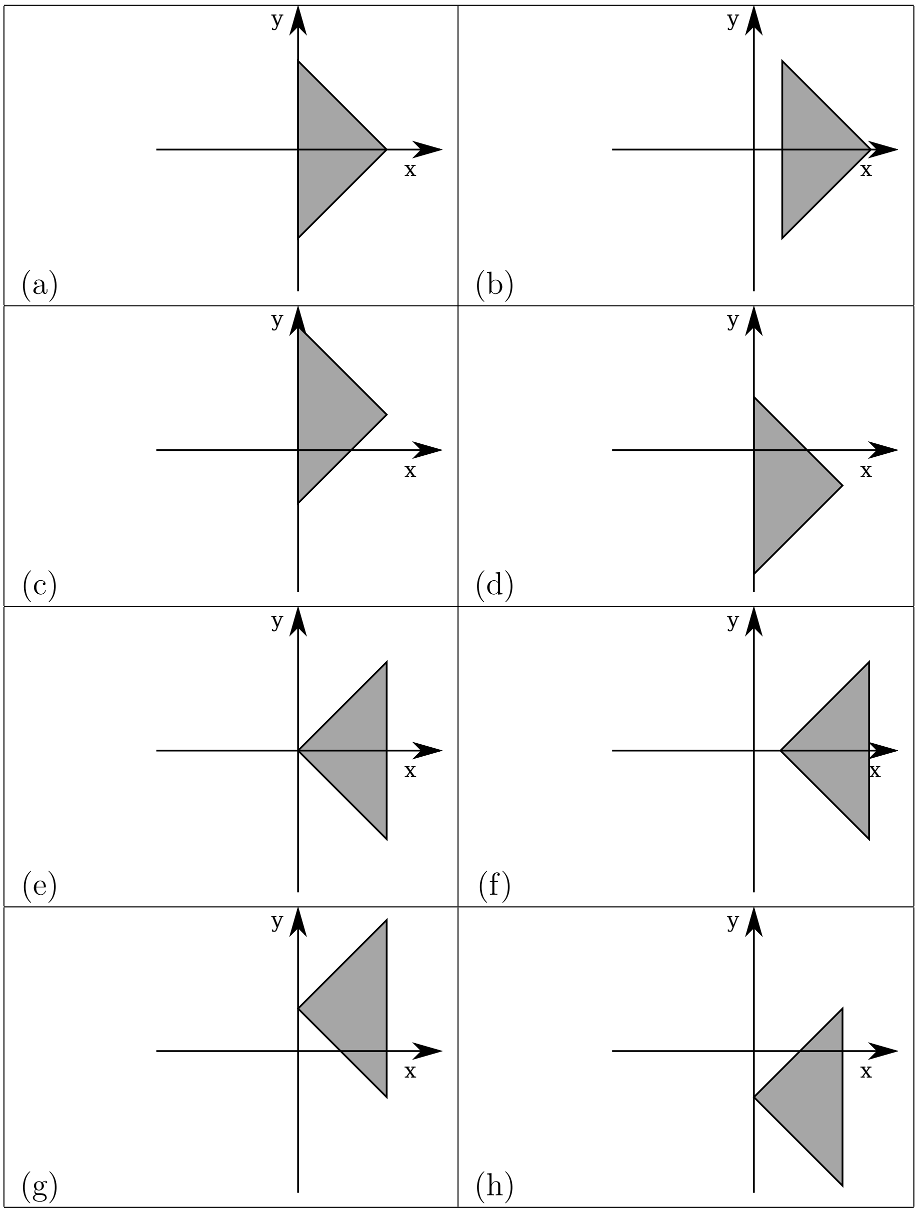
II. True. This is Lemma 2.1 on page 217 of the textbook.

III. True. This is a combination of Theorem 2.1 on page 188 and Corollary 3.2 on page 221 of the textbook.

8. The region, R, for the integral below is an isosceles triangle.

$$\iint_R f(x,y)\,dA = \int_{-3}^{-1}\int_0^{y+3} f(x,y)\,dx\,dy + \int_{-1}^{1}\int_0^{1-y} f(x,y)\,dx\,dy$$

Choose the best triangle below representing the region of integration.



Solution: (d) is correct. This integral could also be written as

$$\int_0^2 \int_{x-3}^{1-x} f(x,y)\,dy\,dx$$

9. Which of the following integrals represents the volume of the solid lying between $x = 1$ and $x = 5 - y^2$, above $z = 0$, and below $z = x^2 + y$.

(a) $\displaystyle\int_0^2 \int_1^{5-x^2} x^2 + y \, dy \, dx$

(b) $\displaystyle\int_0^2 \int_1^{5-y^2} x^2 + y \, dx \, dy$

(c) $\displaystyle\int_{-2}^2 \int_1^{5-x^2} x^2 + y \, dy \, dx$

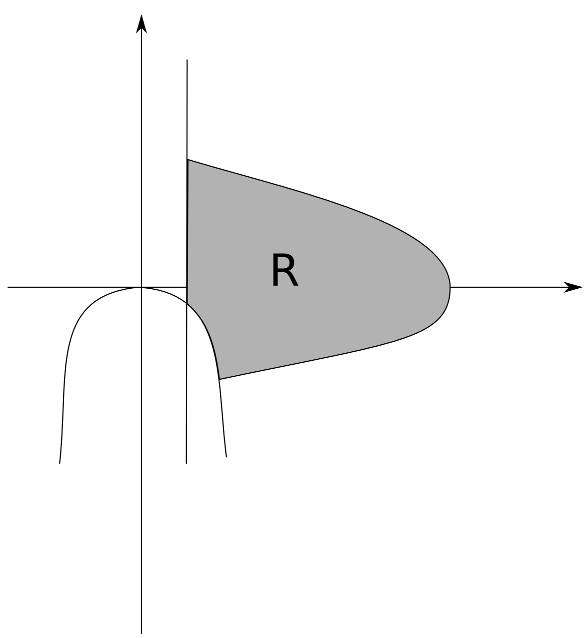
(d) $\displaystyle\int_{-2}^2 \int_1^{5-y^2} x^2 + y \, dx \, dy$ $\longrightarrow$ **CORRECT (Almost Correct, 4 points)**

(e) $\displaystyle\int_0^2 \int_1^{x^2+y} 5 - y^2 \, dy \, dx$

(f) $\displaystyle\int_0^2 \int_1^{x^2+y} 5 - y^2 \, dx \, dy$

(g) None of the above $\longrightarrow$ **CORRECT**

Solution: For the solid in this question, we need to integrate the following region in the xy-plane.



Therefore, none of these answers are correct.

10. The following

$$\int_{-5}^{3}\int_{(1-y)/2}^{\sqrt{4-y}} f(x,y)\;dx\;dy + \int_{3}^{4}\int_{-\sqrt{4-y}}^{\sqrt{4-y}} f(x,y)\;dx\;dy$$

is equivalent to an integral of the form

$$\int_{a}^{b}\int_{A(x)}^{B(x)} f(x,y)\;dy\;dx$$

What is

$$b - a + B(1) - A(1)$$

(a) 5
(b) 6
(c) 7
(d) 8 $\longrightarrow$ **CORRECT**
(e) 9
(f) 10
(g) 11
(h) 12
(i) 13
(j) 14
(k) 15

Solution: The integral is equivalent to

$$\int_{-1}^{3}\int_{-2x+1}^{4-x^2} f(x,y)\;dx\;dy$$

11. Compute

$$\int_0^{\sqrt{\pi}} \int_{y/2}^{\sqrt{\pi}/2} \sin(x^2)\ dx\ dy$$

(a) 1

(b) -1

(c) $\frac{3}{2}$

(d) $\frac{1}{2}$

(e) $1 + \frac{1}{\sqrt{2}}$

(f) $1 - \frac{1}{\sqrt{2}}$ $\longrightarrow$ **CORRECT**

(g) $1 + \frac{\sqrt{3}}{2}$

(h) $1 - \frac{\sqrt{3}}{2}$

Solution: Change the order of integration

$$\begin{aligned}\int_0^{\sqrt{\pi}} \int_{y/2}^{\sqrt{\pi}/2} \sin(x^2)\ dx\ dy &= \int_0^{\sqrt{\pi}/2} \int_0^{2x} \sin(x^2)\ dy\ dx \\ &= \int_0^{\sqrt{\pi}/2} 2x \sin x^2\ dx = 1 - \frac{1}{\sqrt{2}}\end{aligned}$$

12. Which of the following integrals is positive (strictly greater than 0)?

I. $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x\,dy\,dx$

II. $\displaystyle\int_{-3}^{0}\int_{-1}^{7} x\,dx\,dy$

III. $\displaystyle\int_{-3}^{0}\int_{-1}^{7} x\,dy\,dx$

IV. $\displaystyle\int_{0}^{2}\int_{-\sqrt{1-(x-1)^2}}^{\sqrt{1-(x-1)^2}} x\,dy\,dx$

(a) I only
(b) II only
(c) III only
(d) IV only
(e) I and II only
(f) I and III only
(g) I and IV only
(h) II and III only
(i) II and IV only $\longrightarrow$ **CORRECT**
(j) III and IV only
(k) Some other combination or none are positive

Solution: The easiest thing to do is look at the regions and the symmetry of the function we are integrating. Of course, you can compute:

I. $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x\,dy\,dx = 0$

II. $\displaystyle\int_{-3}^{0}\int_{-1}^{7} x\,dx\,dy = 72$

III. $\displaystyle\int_{-3}^{0}\int_{-1}^{7} x\,dy\,dx = -36$

IV. $\displaystyle\int_{0}^{2}\int_{-\sqrt{1-(x-1)^2}}^{\sqrt{1-(x-1)^2}} x\,dy\,dx = \pi$

13. Select the polar form of the equation $x^2 = y^3$.

    (a) $r = \tan\theta$
    (b) $r = (\tan\theta)^{3/2}$
    (c) $r = (\tan\theta)^{2/3}$
    (d) $r = \cos\theta$
    (e) $r = (\cos\theta)^{3/2}$
    (f) $r = (\cos\theta)^{2/3}$
    (g) $r = \sin\theta$
    (h) $r = (\sin\theta)^{3/2}$
    (i) $r = (\sin\theta)^{2/3}$
    (j) $r = \tan^2\theta\sec\theta$
    (k) $r = \cot^2\theta\csc\theta \longrightarrow$ **CORRECT**

Solution: Plug in $x = r\cos\theta$ and $y = r\sin\theta$ and manipulate

$$r^2\cos^2\theta = r^3\sin^3\theta$$
$$r^3\sin^3\theta - r^2\cos^2\theta = 0$$
$$r^2(r\sin^3\theta - \cos^2\theta) = 0$$
$$r = \frac{\cos^2\theta}{\sin^3\theta} = \cot^2\theta\csc\theta$$

14. Compute $\iint_R \frac{1}{(x^2+y^2)^{3/2}}\, dA$ where R is the annulus

$$R = \{(x,y) | 1 \leq x^2 + y^2 \leq 4\}$$

(a) 0

(b) $\frac{\pi}{4}$

(c) $\frac{\pi}{2}$

(d) $\frac{3\pi}{4}$

(e) $\pi \longrightarrow$ **CORRECT**

(f) $\frac{5\pi}{4}$

(g) $\frac{3\pi}{2}$

(h) $\frac{7\pi}{4}$

(i) 2π

Solution: Convert to polar coordinates

$$\begin{aligned}\iint_R \frac{1}{(x^2+y^2)^{3/2}}\, dA &= \int_0^{2\pi} \int_1^2 \frac{1}{r^3}\, r\, dr\, d\theta \\ &= \int_0^{2\pi} \frac{1}{2}\, d\theta = \pi\end{aligned}$$

15. Compute

$$\int_0^{\pi/4}\int_0^{\tan\theta\sec\theta} 5r^2\sin\theta\ dr\ d\theta + \int_{\pi/4}^{\pi/2}\int_0^{\csc\theta} 5r^2\sin\theta\ dr\ d\theta$$

(a) Something less than 0
(b) 1
(c) 2 $\longrightarrow$ **CORRECT**
(d) 3
(e) 4
(f) 5
(g) 6
(h) 7
(i) 8
(j) 9
(k) Something greater than 9

Solution: Convert this integral to rectangular coordinates.

$$\int_0^{\pi/4}\int_0^{\tan\theta\sec\theta} 5r^2\sin\theta\ dr\ d\theta + \int_{\pi/4}^{\pi/2}\int_0^{\csc\theta} 5r^2\sin\theta\ dr\ d\theta = \int_0^1\int_{x^2}^1 5y\ dy\ dx = 2$$

WRITTEN PROBLEM—SHOW YOUR WORK

Name:
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Note: You will be graded on the readability of your work. Use the back of the page, if necessary.

16. Compute the volume of the ice cream cone shaped solid bounded above by the hemisphere: $z = 1 + \sqrt{1 - x^2 - y^2}$ and bounded below by the cone: $z = \sqrt{x^2 + y^2}$.

Solution: Set up the integral over the unit disc and convert to polar

$$\iint_R 1 + \sqrt{1 - x^2 - y^2} - \sqrt{x^2 - y^2}\, dA = \int_0^{2\pi} \int_0^1 (1 + \sqrt{1 - r^2} - r) r\, dr\, d\theta$$
$$= \int_0^{2\pi} \int_0^1 (1 + \sqrt{1 - r^2} - r) r\, dr\, d\theta = \pi$$

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Name:
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Note: You will be graded on the readability of your work. Use the back of the page, if necessary.

17. Let F be the vector field in the plane given by $F(x, y) = (x - y^3, x^3 - y)$. Let R be the half disc defined by $x \geq 0$ and $x^2 + y^2 \leq 1$ Let C be the boundary of R, parametrized positively (counter-clockwise).

   (a) Draw the region R.

   (b) Write down a parametrization of C.

   Solution: C is made up of two curves, $C = C_1 \cup C_2$.

$$\begin{aligned} C_1(t) =& (\cos t, \sin t) \quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \\ C_2(t) =& (0, -t) \quad -1 \leq t \leq 1 \end{aligned}$$

   (c) Write down, using Green's Theorem, a double integral that is equal to

$$\int_C F \cdot dr$$

   Solution:

$$\iint_R 3x^2 + 3y^2 \, dA$$

   (d) Compute, by any means of your choice

$$\int_C F \cdot dr$$

   Solution: We use Green's Theorem and convert to polar coordinates.

$$\int_C F \cdot dr = \iint_R 3x^2 + 3y^2 \, dA = \int_{-\pi/2}^{\pi/2} \int_0^1 3r^3 \, dr \, d\theta = \frac{3\pi}{4}$$