Name:
ID:
Section:

This exam has 17 questions:

- 15 multiple choice questions worth 5 points each.
- 2 hand graded questions worth 25 points total.

Important:

- No graphing calculators! Any non scientific calculator is fine.
- For the multiple choice questions, mark your answer on the answer card.
- Show all your work for the written problems. You will be graded on the ease of reading your solution.
- You are allowed both sides of 3×5 note "cheat card" for the exam.

1. Let

$$A = \{(x, y, z) | x^2 + 7y^2 \leq 1\} \subset \mathbb{R}^3$$

Which of the following statements are true?

I. A is closed
II. A is open
III. A is bounded
IV. A is connected

(a) I and II only
(b) I and III only
(c) I and IV only $\longrightarrow$ **CORRECT**
(d) II and III only
(e) II and IV only
(f) I, II and III only
(g) I, II and IV only
(h) I, III and IV only
(i) II, III and IV only
(j) All are true
(k) None are true

Solution: A is a solid elliptical cylinder. You can view A by drawing the solid ellipse in $\mathbb{R}^2$ and then allowing the ellipse to slide in the z direction forming a solid cylinder.

A is a closed set since it contains all its boundary points. A is unbounded since it contains arbitrarily large z values (in fact, A contains the z-axis). A is connected since any two points can be connected by a curve (in fact, A is convex).

2. Compute the maximal rate of change of $f(x, y, z) = y^2 \cos z \sin x$ at the point (π, π, π).

   (a) $-\pi^2$

   (b) -2π

   (c) $-\pi$

   (d) 0

   (e) π

   (f) 2π

   (g) π^2 $\longrightarrow$ **CORRECT**

   Solution: The maximal rate of change will be equal to the norm of the gradient vector.

   $$\|\nabla f(\pi, \pi, \pi)\| = \|(\pi^2, 0, 0)\| = \pi^2$$

3. Suppose you know

$$\begin{aligned} f(-2,3) =& 7 \\ f_x(-2,3) =& 5 \\ f_y(-2,3) =& 1 \end{aligned}$$

At the point $(-2,3)$, at what rate does the function increase in the direction of the vector $(1,-3)$?

(a) 0
(b) $\frac{1}{\sqrt{10}}$
(c) $\frac{2}{\sqrt{10}}$ $\longrightarrow$ **CORRECT**
(d) $\frac{3}{\sqrt{10}}$
(e) $\frac{4}{\sqrt{10}}$
(f) $\frac{5}{\sqrt{10}}$
(g) $\frac{6}{\sqrt{10}}$
(h) $\frac{7}{\sqrt{10}}$
(i) $\frac{8}{\sqrt{10}}$
(j) $\frac{9}{\sqrt{10}}$
(k) $\frac{10}{\sqrt{10}}$

Solution: This is asking for a directional derivative in the direction of $u = \frac{1}{\sqrt{10}}(1,-3)$.

$$D_u f(-2,3) = u \cdot \nabla f(-2,3) = \frac{1}{\sqrt{10}}(1,-3) \cdot (5,1) = \frac{2}{\sqrt{10}}$$

4. Let $f(x,y) = (3x-y)^2$.

   Use a linear approximation at the point $(1,2)$ to approximate $f(2,3)$.

   (a) 3
   (b) 4
   (c) 5 $\longrightarrow$ **CORRECT**
   (d) 6
   (e) 7
   (f) 8
   (g) 9
   (h) 10
   (i) 11
   (j) 12
   (k) 13

   Solution: The linearization is

$$\begin{aligned} L(x,y) =& f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2) \\ =& 1 + 6(x-1) - 2(y-2) \\ L(2,3) =& 5 \end{aligned}$$

5. Consider the surface defined by the equation

$$x^2z + 3yz^2 + 3xyz = 7$$

The plane that is tangent to this surface at the point $(1, 1, 1)$ can be written in the form

$$5x + By + Cz = D$$

What is $B + C + D$?

(a) 31
(b) 32
(c) 33
(d) 34
(e) 35
(f) 36
(g) 37 $\longrightarrow$ **CORRECT**
(h) 38
(i) 39
(j) 40

Solution: The normal to the tangent plane is $\nabla F(1, 1, 1) = (5, 6, 10)$. Thus, the plane has equation

$$5x + 6y + 10z = 21$$

and $B + C + D = 37$.

6. Let

$$f(x,y) = xy^2 + x^5y^4 - \frac{x}{y^2}$$

Let $T_2(x,y)$ be the the second order Taylor polynomial centered at the point $(1,1)$.

$$T_2(1,1) =$$

(a) 0
(b) 1 $\longrightarrow$ **CORRECT**
(c) 2
(d) 3
(e) 4
(f) 5
(g) 6
(h) 7
(i) 8
(j) 9
(k) 10

Solution: Notice that while you can compute the Taylor polynomial it is much easier to notice that the Taylor polynomial will agree with the function at the point. Thus,

$$T_2(1,1) = f(1,1) = 1$$

7. Let

$$f(x, y) = x^3 y$$

Let $T_2(x, y)$ be the the second order Taylor polynomial centered at the point $(1, 2)$.

$$T_2(0, 2) =$$

(a) 0
(b) 1
(c) 2 $\longrightarrow$ **CORRECT**
(d) 3
(e) 4
(f) 5
(g) 6
(h) 7
(i) 8
(j) 9
(k) 10

Solution:

$$T_2(x, y) = 2 + 6(x-1) + (y-2) + \frac{1}{2}\left[12(x-1)^2 + 2(3)(x-1)(y-2)\right]$$
$$T_2(0, 2) = 2$$

8. Suppose $f(x, y)$ attains a maximum value of 17 at the point $(3, 4)$.

   Which of the following is guaranteed to be a tangent plane to the surface?

   (a) $x = 3$
   (b) $y = 4$
   (c) $z = 17$ $\longrightarrow$ **CORRECT**
   (d) $z = (x - 3) + (y - 4) + 17$
   (e) $(x - 3) + (y - 4) + 17z = 0$
   (f) None of the above are guaranteed to be tangent to the surface. $\longrightarrow$ **CORRECT**

   Solution: Since there is a maximum of 17 there must be a horizontal tangent plane at that point, and this plane must have equation $z = 17$.

   (f) can also be considered correct. If the function is not differentiable or if the maximum occurs on the boundary of the domain then there would not be a tangent plane.

9. Which of the following statements about a smooth function $f(x, y)$ must always be true?

   I. If the tangent plane to f at a point p is parallel to the plane $z = 0$, then f has either a local maximum or minimum at the point p.

   II. If U is an open subset of the plane, and f has no critical points in U, then f has no maximum or minimum in U.

   III. If C is a closed subset of the plane, and f has no critical points on C, then f attains a maximum or minimum on the boundary of C.

   (a) I only
   (b) II only $\longrightarrow$ **CORRECT**
   (c) III only
   (d) I and II only
   (e) I and III only
   (f) II and III only
   (g) I, II, and III only
   (h) None must be true

   Solution:

   I. Cannot be true since the point p can be a saddle point.

   II. This is true–a maximum or minimum would have to occur at a critical point (note, there are no boundary points in U since U is open).

   III. May not be true, the set C can be unbounded and have no maximum or minimum.

10. Suppose you have a function $g(x, y)$ and you know the following

$$\begin{aligned} g(3,7) =& 12 \\ g_x(3,7) =& 0 \\ g_y(3,7) =& 0 \\ g_{xx}(3,7) =& -8 \\ g_{xy}(3,7) =& -6 \\ g_{yy}(3,7) =& -6 \end{aligned}$$

Which must be true

(a) g has a local maximum at $(3, 7)$ $\longrightarrow$ **CORRECT**
(b) g has a local minimum at $(3, 7)$
(c) g has a saddle point at $(3, 7)$
(d) The point $(3, 7)$ is neither a local max, local min nor saddle.
(e) More than one of the above are true
(f) None of the above are true

Solution: Since $g_x(3, 7) = 0$ and $g_y(3, 7) = 0$ the point $(3, 7)$ is a critical point. We use the second derivative test.

$$D = g_{xx}(P)g_{yy}(P) - (g_{xy}(P))^2 = (-8)(-6) - (-6)^2 = 12$$

Thus we either have a local max or a local min at $(0, 0)$. Since $g_{xx}(P) = -8$ we must have a local maximum.

11. Let

$$f(x,y) = y^4 - y^2 + x^2 - 2xy$$

The function f has how many local maxima, local minima, and saddle points?

(a) 1 max, 2 min, 0 saddle
(b) 2 max, 1 min, 0 saddle
(c) 1 max, 1 min, 1 saddle
(d) 0 max, 2 min, 1 saddle $\longrightarrow$ **CORRECT**
(e) 2 max, 0 min, 1 saddle
(f) 2 max, 2 min, 1 saddle
(g) 0 max, 4 min, 1 saddle
(h) 4 max, 0 min, 1 saddle

Solution: The critical points are at $(0,0)$, $(1,1)$ and $(-1,-1)$. We can test these

P	$f(P)$	D	$f_{xx}(P)$	Conclusion
$(-1,-1)$	-1	16	2	Local Min
$(0,0)$	0	-8	2	Saddle
$(1,1)$	-1	16	2	Local Min

12. Find the minimal distance from the origin to the curve $3x^2 + 2xy + 3y^2 = 1$.

    (a) 0
    (b) $\frac{1}{2\sqrt{2}}$
    (c) $\frac{1}{4}$
    (d) $\frac{1}{2}$ $\longrightarrow$ **CORRECT**
    (e) $\frac{1}{\sqrt{2}}$
    (f) 1
    (g) $\sqrt{2}$

Solution: Minimize $f(x, y) = x^2 + y^2$ subject to $3x^2 + 2xy + 3y^2 = 1$. This is a good problem for Lagrange multipliers which gives you equations

$$\begin{aligned} 2x =& \lambda(6x + 2y) \\ 2y =& \lambda(2x + 6y) \\ 3x^2 + 2xy + 3y^2 =& 1 \end{aligned}$$

Solving this system gives

λ	(x, y)	$f(x, y)$	Conclusion
$\frac{1}{2}$	$(-1/2, 1/2)$	$\frac{1}{2}$	Max, Dist $= 1/\sqrt{2}$
$\frac{1}{2}$	$(1/2, -1/2)$	$\frac{1}{2}$	Max, Dist $= 1/\sqrt{2}$
$\frac{1}{4}$	$\left(\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$	$\frac{1}{4}$	Min, Dist $= 1/2$
$\frac{1}{4}$	$\left(-\frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right)$	$\frac{1}{4}$	Min, Dist $= 1/2$

13. Let F be the vector field.

$$F = (2x\cos y, e^y - x^2 \sin y)$$

Let ϕ be a potential function such that $\phi(0,0) = 1$. Find $\phi(1,0)$.

(a) 0
(b) 1
(c) 2 $\longrightarrow$ **CORRECT**
(d) 3
(e) 4
(f) 5
(g) 6
(h) 7
(i) 8
(j) 9
(k) The vector field F has no potential function.

Solution: $\phi = x^2 \cos y + e^y$.

14. Let F be the vector field.

$$F = (3x^2y, x^3 + 2yz, y^2 + x)$$

Let ϕ be a potential function such that $\phi(0,0) = 1$. Find $\phi(1,1)$.

CORRECTION: Let ϕ be a potential function such that $\phi(0,0,0) = 1$. Find $\phi(1,1,1)$.

(a) 0
(b) 1
(c) 2
(d) 3
(e) 4
(f) 5
(g) 6
(h) 7
(i) 8
(j) 9
(k) The vector field F has no potential function. $\longrightarrow$ **CORRECT**

Solution: You can compute the curl of the vector field

$$\text{curl } F = (0, -1, 0) \neq (0, 0, 0)$$

and therefore there is not a potential function.

15. Let F be the vector field

$$F = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$

Which of the following are true?

I. F has a potential function definied on the entire plane, $\mathbb{R}^2$.

II. F has a potential function definied on the right half plane, $x > 0$.

III. F has a potential function definied on the punctured plane, $(x, y) \neq (0, 0)$.

IV. F has no potential function.

(a) I only
(b) II only $\longrightarrow$ **CORRECT**
(c) III only
(d) IV only
(e) I and II only
(f) I and III only
(g) II and III only
(h) I, II and III only
(i) All are true
(j) None are true

Solution: This is discussed in section 7.3 of the text. The point is that the potential function can only be defined on a rectangle and only answer II is a rectangle (albeit an infinite one).

WRITTEN PROBLEM—SHOW YOUR WORK

Name:
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Note: You will be graded on the readability of your work. Use the back of the page, if necessary.

16. Find the maximum and minimum of the function $f(x, y) = xy$ on the domain $0 \leq y \leq 1 - x^2$? Be sure to state where the maximum and minimum occur.

Solution: First, we find critical points inside the boundary: $\nabla f = (y, x)$ which give the only critical point as $(0, 0)$.

Then, we look for maximum and minima on the boundary of the domain.

Along the parabola we have $r(t) = (t, 1 - t^2)$, $-1 \leq t \leq 1$. $f(r(t)) = t(1 - t^2)$ and gives the critical points at $t = \pm 1/\sqrt{3}$. Thus, we have critical points $(1/\sqrt{3}, 2/3)$ and $(-1/\sqrt{3}, 2/3)$.

Along the bottom of the region we have $y = 0$ and $-1 \leq x \leq 1$. $f(x, 0) = 0$ and thus the max and min along that portion of the boundary are both 0.

(x, y)	$f(x, y)$	Conclusion
$(0, 0)$	0	
$(x, 0)$	0	
$(-1/\sqrt{3}, 2/3)$	$-\frac{2}{3\sqrt{3}}$	Min
$(1/\sqrt{3}, 2/3)$	$\frac{2}{3\sqrt{3}}$	Max

WRITTEN PROBLEM—SHOW YOUR WORK

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17. Let

$$f(x, y) = xy^2 - \frac{1}{2}y^4 - \frac{1}{3}x^3$$

(a) Find all critical points of f.

(b) For each critical point of f, determine everything the second derivative test tells you about that critical point.

Solution:

(x, y)	$f(x, y)$	D	$f_{xx}(x, y)$	Conclusion
$(0, 0)$	0	0	0	2nd deriv test says nothing about this point
$(1, -1)$	1/6	4	-2	Local max
$(1, 1)$	1/6	4	-2	Local max

Incidentally (but this wasn't part of the question), the point $(0, 0)$ is a saddle point which you can see by composing the function f with the curves $r_1(t) = (t, 0)$ and $r_2(t) = (0, t)$.