1. If $v_1$ and $v_2$ are vectors in $\mathbb{R}^3$ with $\|v_1\| = 2$, $\|v_2\| = 5$ and $v_1 \cdot v_2 = -6$ find $\|v_1 \times v_2\|$.

(a) 0
(b) $2\sqrt{3}$
(c) $3\sqrt{2}$
(d) $\sqrt{51}$
(e) 8 — CORRECT
(f) $4\sqrt{5}$
(g) $5\sqrt{3}$
(h) $2\sqrt{21}$
(i) 10
(j) None of the above

**Solution:** Let $\theta$ be the angle between $v_1$ and $v_2$. Since $v_1 \cdot v_2 = -6$ we have

$$\cos \theta = \frac{v_1 \cdot v_2}{\|v_1\|\|v_2\|} = \frac{-6}{(2)(5)} = \frac{-3}{5}$$

$$\|v_1 \times v_2\| = \|v_1\|\|v_2\|\sin \theta = \|v_1\|\|v_2\|\sqrt{1 - \cos^2 \theta}$$

$$= (2)(5)\sqrt{1 - (-3/5)^2} = 8$$
2. Which of the following is an equation of the plane containing the origin and parallel to the two vectors

\[ v_1 = (1, 3, 4), \quad v_2 = (2, -1, 1) \]

(a) \( x + 3y + 4z = 0 \)
(b) \( 2x - y + z = 0 \)
(c) \( x + y - z = 0 \) \( \rightarrow \) CORRECT
(d) \( x - y - z = 0 \)
(e) \( x + 3y + 4z = 3 \)
(f) \( 2x - y + z = 3 \)
(g) \( x - y - z = 3 \)
(h) None of the above

Solution: A normal vector for the plane is

\[ v_1 \times v_2 = (7, 7, -7) \]

Or, alternatively, you can solve for \( N = (a, b, c) \) in the equations \( N \cdot v_1 = 0 \) and \( N \cdot v_2 = 0 \).

Since the vector \( N = (1, 1, -1) \) is parallel to the vector above, we can just use this vector for the normal vector, arriving at the equation:

\[ x + y - z = D \]

Using the point on the plane, \( (0, 0, 0) \), we find that \( D = 0 \).
3. Which of the following are equations for a plane that is parallel to the line 
\[ r(t) = (1 + t, 2 - t, 3 + 2t) \]

(a) \( x + y = 6 \)  
(b) \( x + y = 5 \)  
(c) \( 2y + z = 6 \)  
(d) \( 2y + z = 5 \)  
(e) \( 2x + 4y + z = 0 \)  
(f) All of these \( \rightarrow \) CORRECT  
(g) More than one of these, but not zero or all of them  
(h) None of these  

Solution: The direction vector of the given line is \( v = (1, -1, 2) \). The line will be parallel to the plane if the vector \( v \) is normal to the normal vector of the plane. So, we just check dot products of the normal vectors of the planes:

<table>
<thead>
<tr>
<th>Plane</th>
<th>( N ) for plane</th>
<th>( N \cdot v )</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x + y = 6 )</td>
<td>( (1, 1, 0) )</td>
<td>0</td>
<td>Line is parallel to plane</td>
</tr>
<tr>
<td>( x + y = 5 )</td>
<td>( (1, 1, 0) )</td>
<td>0</td>
<td>Line is parallel to plane</td>
</tr>
<tr>
<td>( 2y + z = 6 )</td>
<td>( (0, 2, 1) )</td>
<td>0</td>
<td>Line is parallel to plane</td>
</tr>
<tr>
<td>( 2y + z = 5 )</td>
<td>( (0, 2, 1) )</td>
<td>0</td>
<td>Line is parallel to plane</td>
</tr>
<tr>
<td>( 2x + 4y + z = 0 )</td>
<td>( (2, 4, 1) )</td>
<td>0</td>
<td>Line is parallel to plane</td>
</tr>
</tbody>
</table>
4. Find the distance from the point \((3, 1, 1)\) to the line \(r(t) = (2 - t, 2 + t, 1 + 2t)\).

(a) 0  
(b) \(\frac{1}{\sqrt{6}}\)  
(c) \(\frac{1}{\sqrt{3}}\)  
(d) \(\frac{2}{\sqrt{3}}\) \(\rightarrow\) CORRECT  
(e) 2  
(f) 3  
(g) \(\sqrt{6}\)  
(h) \(2\sqrt{5}\)  
(i) \(\sqrt{11}\)  
(j) None of the above

**Solution:** Note that \(r(t) = (2, 2, 1) + t(-1, 1, 2)\). Then we have to use vector projections

\[
Q = (2, 2, 1) \quad \text{Point on line}
\]
\[
v = (-1, 1, 2) \quad \text{Direction vector of the line}
\]
\[
P = (3, 1, 1) \quad \text{The point}
\]
\[
\overrightarrow{QP} = (1, -1, 0) \quad \text{Vector from } Q \text{ to } P
\]
\[
c = \frac{v \cdot \overrightarrow{QP}}{v \cdot v} \quad \text{Component of } \overrightarrow{QP} \text{ along } v
\]
\[
cv = \frac{v \cdot \overrightarrow{QP}}{v \cdot v} v = \left(\frac{1}{3}, -\frac{1}{3}, -\frac{23}{3}\right)
\]
\[
\|\overrightarrow{QP} - cv\| = \left\|\left(\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}\right)\right\| = \frac{2}{\sqrt{3}}
\]
5. Find the distance between the planes

\[ x - 2y - z = 1 \]
\[ x - 2y - z = 4 \]

(a) 0
(b) \( \frac{1}{\sqrt{6}} \)
(c) \( \frac{1}{\sqrt{3}} \)
(d) \( \frac{1}{\sqrt{2}} \)
(e) \( \sqrt{\frac{2}{3}} \)
(f) \( \sqrt{3} \) → CORRECT
(g) \( \sqrt{2} \)
(h) \( \sqrt{3} \)
(i) \( \sqrt{6} \)
(j) None of the above

Solution: The only real difference between this problem and finding the distance from a point to a plane is that we have to select two arbitrary points on the two planes. So, let \( Q = (1, 0, 0) \) be on the first plane and let \( P = (4, 0, 0) \) be on the second plane. Then, \( \overrightarrow{QP} = (3, 0, 0) \) and \( N = (1, -2, -1) \) is the normal vector of the planes.

\[
\| \text{Pr}_N \overrightarrow{QP} \| = \left\| \frac{N \cdot \overrightarrow{QP}}{N \cdot N} N \right\| = \left\| \left( -\frac{1}{2}, 1, \frac{1}{2} \right) \right\| = \sqrt{\frac{3}{2}}
\]
6. Find an equation of the line through the point \((1, 0, 2)\) that is orthogonal to the plane

\[ x + 2y + 7z = 12 \]

(a) \(r(t) = (1 + t, 2, 7 + 2t)\)
(b) \(r(t) = (1 + t, 2t, 2 + 7t)\) \(\rightarrow\) CORRECT
(c) \(r(t) = (1 - t, 2t, 2 - 7t)\)
(d) \(r(t) = (12 + t, 12, 12 + 2t)\)
(e) \(r(t) = t + 2t + 7t - 12\)
(f) \(r(t) = (x + 1, 2y, 7z + 2)\)
(g) None of the above

**Solution:** To be orthogonal to the plane means the direction vector of the plane is \(v = (1, 2, 7)\). Thus, the equation of the line is

\[ r(t) = (1, 0, 2) + t(1, 2, 7) \]
7. Compute the larger angle between the two planes $x + z = 213$ and $x + y = -17$.

(a) 0
(b) $\pi/8$
(c) $\pi/6$
(d) $\pi/5$
(e) $\pi/4$
(f) $\pi/2$
(g) $2\pi/3$ $\rightarrow$ CORRECT
(h) $3\pi/4$
(i) $5\pi/6$
(j) $\pi$

Solution: Here we find the angle between normal vectors of the planes:

$N_1 = (1, 0, 1)$

$N_2 = (1, 1, 0)$

$\cos \theta = \frac{N_1 \cdot N_2}{\|N_1\| \|N_2\|} = \frac{1}{2}$

Thus $\theta = \pi/3$ but there are two possible angles, both supplementary. Thus, the larger angle is $2\pi/3$. 
If
\[ r(t) = \left(1 - t, \frac{1}{t}, 2t, -3\right) \quad \text{and} \quad s(t) = (t, -t, t^2, t^3) \]
compute
\[ \frac{d}{dt}(r \cdot s)(t) \bigg|_{t=1} = \]
(a) $-6$
(b) $-5$
(c) $-4 \quad \rightarrow \text{CORRECT}$
(d) $-3$
(e) $-2$
(f) $-1$
(g) $0$
(h) $1$
(i) $2$
(j) $3$
(k) None of the above

**Solution:** You can just multiply out the dot product and then take the derivative. Or, just use the product rule for dot products:

\[
\frac{d}{dt}(r(t) \cdot s(t)) = r'(t) \cdot s(t) + r(t) \cdot s'(t)
\]
\[
= \left(-1, -\frac{1}{t^2}, 2, 0\right) \cdot (t, -t, t^2, t^3) + \left(1 - t, \frac{1}{t}, 2t, -3\right) \cdot (1, -1, 2t, 3t^2)
\]
\[
= -3t^2 - 2t + 1
\]
\[
\left. \frac{d}{dt}(r(t) \cdot s(t)) \right|_{t=1} = -4
\]
9. Find the equation of the line that is tangent to the curve

\[ r(t) = \left( t, t^2, \frac{2}{t} \right) \]

when \( t = 2 \).

(a) \( L(t) = (1, 1, 2) + t(1, 2, -2) \)
(b) \( L(t) = (1, 2 - 2) + t(1, 1, 2) \)
(c) \( L(t) = (1, 1, 2) + t(0, 2, 4) \)
(d) \( L(t) = (0, 2, 4) + t(1, 1, 2) \)
(e) \( L(t) = (-2, 4, -1) + t\left(1, -4, -\frac{1}{2}\right) \)
(f) \( L(t) = (1, -4, -\frac{1}{2}) + t(-2, 4, -1) \)
(g) \( L(t) = (1, 4, -\frac{1}{2}) + t(2, 4, 1) \)
(h) \( L(t) = (2, 4, 1) + t\left(1, 4, -\frac{1}{2}\right) \rightarrow \text{CORRECT} \)
(i) None of the above

Solution:

\[ r(2) = (2, 4, 1) \]
\[ r'(t) = \left( 1, 2t, -\frac{2}{t^2} \right) \]
\[ r'(2) = \left( 1, 4, -\frac{1}{2} \right) \]
\[ L(t) = (2, 4, 1) + t\left(1, 4, -\frac{1}{2}\right) \]
10. Find a rectangular representation for the parametrized curve 

\[ r(t) = (19 - 4 \sin 3t, 23 + 3 \cos 3t) \]

(a) \( \left( \frac{x - 19}{4} \right)^2 + \left( \frac{y - 23}{3} \right)^2 = 1 \) → CORRECT
(b) \( \left( \frac{x + 19}{4} \right)^2 + \left( \frac{y + 23}{3} \right)^2 = 1 \)
(c) \( \left( \frac{x - 23}{4} \right)^2 + \left( \frac{y - 19}{3} \right)^2 = 1 \)
(d) \( \left( \frac{x + 23}{4} \right)^2 + \left( \frac{y + 19}{3} \right)^2 = 1 \)
(e) \( (x - 19)^2 + (y - 23)^2 = 1 \)
(f) \( (x + 19)^2 + (y + 23)^2 = 1 \)
(g) \( (x - 23)^2 + (y - 19)^2 = 1 \)
(h) \( (x + 23)^2 + (y + 19)^2 = 1 \)
(i) \( (x - 19)^2 + (y - 23)^2 = 12 \)
(j) \( (x + 19)^2 + (y + 23)^2 = 12 \)
(k) None of the above

**Solution:** This is an ellipse, with center at \((19, 23)\). There are several ways to see the correct answer but is straightforward to substitute \( r(t) \) into the equations and see that in the equation in the marked correct answer is satisfied.
11. Let $f(x,y,z) = x^2 ye^{-3z}$. Find
\[
\frac{\partial^3 f}{\partial x \partial y \partial z}(3, 2, 0)
\]
(a) $-54$
(b) $-36$
(c) $-27$
(d) $-18 \rightarrow \text{CORRECT}$
(e) 0
(f) 4
(g) 9
(h) 12
(i) 18
(j) 162
(k) None of the above

Solution:

\[
\begin{align*}
\frac{\partial f}{\partial z} &= -3x^2 ye^{-3z} \\
\frac{\partial^2 f}{\partial y \partial z} &= -3x^2 e^{-3z} \\
\frac{\partial^3 f}{\partial x \partial y \partial z} &= -6xe^{-3z} \\
\frac{\partial^3 f}{\partial x \partial y \partial z}(3, 2, 0) &= -18
\end{align*}
\]
12. If \( f(x, y, z) = x^2 - 2xy \ln z \), compute

\[
\nabla f(1, -1, 1) = \text{grad} f(1, -1, 1) =
\]

(a) 0  
(b) 1  
(c) 2  
(d) 4  
(e) \((1, -1, 1)\)  
(f) \((2, 0, 2)\) \(\rightarrow\) CORRECT  
(g) \((2, 0, -2)\)  
(h) \((2, 1, 2)\)  
(i) \((2, 1, -2)\)  
(j) None of the above

**Solution:**

\[
\nabla f = \left(2x - 2y \ln z, -2x \ln z, -\frac{2xy}{z}\right)
\]

\[
\nabla f(1, -1, 1) = (2, 0, 2)
\]
13. Let \( f(x, y, z) = x + y^2 + z^2 \) and \( r(t) = (2 - t, \cos t, \sin t) \). Compute

\[
\left. \frac{d}{dt} (f \circ r)(t) \right|_{t=1} =
\]

(a) \(-2\)
(b) \(-1\) \(\rightarrow\) CORRECT
(c) 0
(d) 1
(e) 2
(f) 3
(g) 4
(h) 5
(i) 6
(j) 7
(k) None of the above

**Solution:** You can substitute and just compute, or you can use the chain rule:

\[
\nabla f = (1, 2y, 2z)
\]

\[
r'(t) = (-1, -\sin t, \cos t)
\]

\[
\frac{d}{dt} (f \circ r(t)) = \nabla f(r(t)) \cdot r'(t) = (-1, 2\cos t, 2\sin t) \cdot (-1, -\sin t, \cos t)
\]

\[
= -1 - 2 - 2 \cos t \sin t + 2 \cos t \sin t = -1
\]

\[
\left. \frac{d}{dt} (f \circ r(t)) \right|_{t=1} = -1
\]
14. Match the graph to the correct function.

(a) \( f(x, y) = y^2 \)
(b) \( f(x, y) = x^2 \)
(c) \( f(x, y) = x - y^2 \)
(d) \( f(x, y) = y - x^2 \)
(e) \( f(x, y) = y^3 - x^2 \) \( \longrightarrow \) CORRECT
(f) \( f(x, y) = x^2 - y^3 \)
(g) \( f(x, y) = y^4 - x^2 \)
(h) \( f(x, y) = x^4 - y^2 \)
(i) \( f(x, y) = y^4 - x \)
(j) \( f(x, y) = x^4 - y \)
15. Match the level curves to the correct function.

(a) \( f(x, y) = x^2 \)
(b) \( f(x, y) = y^2 \)
(c) \( f(x, y) = x^2 - y^2 \)
(d) \( f(x, y) = x^2 + y^2 \)
(e) \( f(x, y) = x + y^2 \)
(f) \( f(x, y) = y + x^2 \)
(g) \( f(x, y) = y + x^2 + 3x \)
(h) \( f(x, y) = y + x^2 - 3x \)
(i) \( f(x, y) = x + y^2 + 3y \)
(j) \( f(x, y) = x + y^2 - 3y \) → CORRECT
16. Consider the curve \( r(t) = (2t^3, 3t^2) \).

(a) Compute the velocity at time \( t \)

(b) Compute the acceleration at time \( t \)

(c) Compute the speed at time \( t \)

(d) Compute the arc length from \( t = 1 \) to \( t = 4 \)

Solution:

\[
\dot{r}(t) = (6t^2, 6t) \quad \text{velocity}
\]

\[
\ddot{r}(t) = (12t, 6) \quad \text{acceleration}
\]

\[
\|\dot{r}(t)\| = \sqrt{36t^4 + 36t^2}
\]

\[
\text{Len} = \int_1^4 \|\dot{r}(t)\| \, dt = \int_1^4 \sqrt{36t^4 + 36t^2} \, dt
\]

\[
= \int_1^4 6t\sqrt{t^2 + 1} \, dt \quad (\text{Let } u = t^2 + 1)
\]

\[
= \int_1^4 3\sqrt{u} \, du = 2u^{3/2}\bigg|_{t=1}^{t=4}
\]

\[
= 2(t^2 + 4)^{3/2}\bigg|_{t=1}^{t=4} = 34\sqrt{17} - 4\sqrt{2}
\]
17. Consider the surface \( z^2 = x^2 - y^2 \)

(a) Draw the slices for \( x = -2, -1, 0, 1, 2 \) in the same plane

Solution:

(b) Draw the slices for \( y = -2, -1, 0, 1, 2 \) in the same plane

Solution:

(c) Draw the slices for \( z = -2, -1, 0, 1, 2 \) in the same plane Solution:
(d) Draw the surface in 3-space.

Solution: