

Math 233 - December 7, 2009

Solutions

- Final Exam: Dec 14, 10:30AM. 1.1-11.2.

Chapter 1: Vectors, lines, planes, vector products
Chapter 2: Curves, length of curves
Chapter 3: Graphing in $\mathbb{R}^3$, level curves, partial derivatives
Chapter 4: Chain rule, tangent planes, directional derivatives
Chapter 5: Critical points, extrema on closed and bounded domains, Lagrange
Chapter 6: Taylor polynomials, second derivative test
Chapter 7: Vector fields, potential functions, special vector field
Chapter 8: Curve integrals, potential functions, dependence on path
Chapter 9: Double integrals, polar coordinates
Chapter 10: Green's Theorem
Chapter 11: Triple integrals, cylindrical and spherical coordinates

1. Find the second degree Taylor polynomial at the point $(1, 2)$ for the function $f(x, y) = x^2y + y$.

Solution: $4 + 2(y - 2) + 4(x - 1) + \frac{1}{2}(4(x - 1)^2 + 4(x - 1)(y - 1))$

2. Use the second degree Taylor polynomial at the point $(1, 2)$ for the function $f(x, y) = x^2y + y$ to approximate $f(2, 0)$.

Solution: $f(2, 0) \approx T_2(2, 0) = 2$

3. Find an equation for the plane through the origin and parallel to the plane $2x - y + 3z = 14$.

Solution: $2x - y + 3z = 0$

4. Find and classify all critical points of the function $f(x, y) = x^4 + y^4 - 4xy + 2$.

Solution: Saddle at $(0, 0)$, min at $(1, 1)$, min at $(-1, -1)$.

5. Let R be the region between the curve $y = \sqrt{x}$ and the x -axis for $0 \leq x \leq 1$. Find $\iint_R \frac{2y}{x^2+1} dA$.

Solution: $(\ln 2)/2$

6. Find the volume under the cone $z = \sqrt{x^2 + y^2}$ and above the disk $x^2 + y^2 \leq 4$.

Solution: $16\pi/3$

7. Compute $\iiint_R x^2 + y^2 + z^2 dV$ where R is the unit ball $x^2 + y^2 + z^2 \leq 1$.

Solution: $4\pi/5$

8. Let $r(t) = (t^3, t, t^2)$, $0 \leq t \leq 1$. Compute $\int_r x^2y\sqrt{z} dz$.

Solution: $1/5$

9. Let C be a curve from $(1, 0, -2)$ to $(4, 6, 3)$. Let $f(x, y, z) = xyz + z^2$. Compute $\int_C \nabla f \cdot dr$.

Solution: 77

10. Let C be the sides of the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, $(0, 1)$ oriented counter-clockwise. Find $\int_C e^y dx + 2xe^y dy$.

Solution: $e - 1$

11. Let $f(x, y, z) = x^2 + 2y^3z$. Find the direction f is decreasing most rapidly at the point $(0, 1, 0)$

Solution: $(0, 0, -1)$

12. Find where the tangent plane to the surface $x^2 + y^3 + z^4 = 10$ at the point $(3, 0, 1)$ hits the z -axis.

Solution: $(0, 0, 11/2)$

13. Find $\iint_R x dA$ where R is the interior of the triangle with vertices $(0, 0)$, $(0, 2)$ and $(1, 2)$.

Solution: $1/3$

14. Find the volume of the solid above the plane $z = 1$, below the surface $z = 2 + x^2 + y^2$ and enclosed by $x^2 + y^2 = 1$.

Solution: $3\pi/2$

15. Find the projection of $(3, 1, 7)$ onto $(-2, 2, 1)$.

Solution: $\frac{1}{3}(-2, 2, 1)$

16. Find the volume under the graph of $f(x, y) = xy$ and above the triangle in the xy -plane with vertices $(0, 1, 0)$, $(2, 1, 0)$ and $(2, 2, 0)$.

Solution: $11/6$

17. Find the average value of $f(x, y, z) = x^2$ over the region bounded by a sphere of radius 1 around the origin.

Solution: $1/5$

18. True/False

(a) Suppose C is the positively oriented boundary of the region R . Then the area $A = \int_C -y dx$.

Solution: True

(b) Suppose $P(x, y)$ and $Q(x, y)$ are functions defined in a region and, in that region we have $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. Then, for any closed loop C we have $\int_C P dx + Q dy = 0$. **Solution:** False (need R to be a rectangle).

(c) Suppose $f(x, y, z)$ has a local maximum at the point (x_0, y_0, z_0) in the interior of its domain. Then, $\nabla f(x_0, y_0, z_0) = 0$. **Solution:** True

(d) Suppose (x_0, y_0, z_0) is in the interior of the domain of f and $\nabla f(x_0, y_0, z_0) = 0$. Then f has either a maximum or minimum at the point (x_0, y_0, z_0) . **Solution:** False

(e) Suppose $f(x, y)$ and $g(x, y)$ both have the same domain and $\nabla f = \nabla g$ at all points in the domain. Then $f(x, y) = g(x, y)$ for all points in the domain. **Solution:** False

(f) Suppose $f(x, y)$ and $g(x, y)$ both have the same domain and $\nabla f = \nabla g$ at all points in the domain. Then there is some constant C so that $f(x, y) = g(x, y) + C$ for all points in the domain. **Solution:** False

19. Find $\int_C x^2 z \, ds$ where C is the line segment from $(0, 1, -1)$ to $(-2, 3, 0)$.

Solution: -1

20. Let C be the closed curve consisting of the line segment from $(-1, 0)$ to $(1, 0)$ and then the upper half of the unit circle back to $(-1, 0)$. Find $\int_C y^2 \cos x \, dx + (2xy + 2y \sin x) \, dy$.

Solution: $4/3$

21. Find the curl of $F = (z^2, zy, x^2 - z)$.

Solution: $(0, -y, 2z - 2x)$

22. Find the center and radius of the sphere $x^2 + y^2 + z^2 - 6x + 2y + 6 = 0$

Solution: $(3, -1, 0)$, Radius 2.

23. Find the arc length of the curve $r(t) = (2t\sqrt{t}, \cos 3t, \sin 3t), 0 \leq t \leq 3$.

Solution: 14

24. Compute $\int_0^4 \int_{\sqrt{y}}^2 \sqrt{1+x^3} \, dx \, dy$.

Solution: $\frac{52}{9}$

25. True/False

(a) The vectors $(1, 3, -4)$ and $(1, 3, 4)$ are orthogonal. **Solution:** False

(b) The vector $(1, 0, 5)$ is longer than the vector $(3, -3, 1)$ **Solution:** True

(c) $(1, 0, 1) \times (2, 3, 3)$ is parallel to $(3, 1, -3)$ **Solution:** True

26. Let $r(t) = (t^3, t, t^4)$, $0 \leq t \leq 1$. Compute $\int_C (3x + 8yz) \, ds$.

Solution: $\frac{1}{18}(26\sqrt{26} - 1)$

27. Let R be the rectangle $[2, 4] \times [0, 1]$. Find a constant M so that $\iint_R x e^y \, dA = M \cdot \text{Area}(R)$.

Solution: $3(e - 1)$

28. Find the volume of the solid bounded by $x^2 + y^2 = 6$ and the planes $y + z = 7$ and $y + z = 14$.

Solution: 42π

29. Find and analyze all critical points of $f(x, y) = 5x^2y + \frac{5}{3}y^3 - 5x^2 - 5y^2 + 7$.

Solution: Max at $(0, 0)$, min at $(0, 2)$, saddles at $(1, 1)$ and $(-1, 1)$.