## Math 233 - December 7, 2009

- Final Exam: Dec 14, 10:30AM. 1.1-11.2.

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|---|---|
| Chapter 1: | Vectors, lines, planes, vector products |
| Chapter 2: | Curves, length of curves |
| Chapter 3: | Graphing in $\mathbb{R}^3$, level curves, partial derivatives |
| Chapter 4: | Chain rule, tangent planes, directional derivatives |
| Chapter 5: | Critical points, extrema on closed and bounded domains, Lagrange |
| Chapter 6: | Taylor polynomials, second derivative test |
| Chapter 7: | Vector fiels, potential functions, special vector field |
| Chapter 8: | Curve integrals, potential functions, dependence on path |
| Chapter 9: | Double integrals, polar coordinates |
| Chapter 10: | Green's Theorem |
| Chapter 11: | Triple integrals, cylindrical and spherical coordinates |

1. Find the second degree Taylor polynomial at the point $(1, 2)$ for the function $f(x, y) = x^2y + y$.
2. Use the second degree Taylor polynomial at the point $(1, 2)$ for the function $f(x, y) = x^2y + y$ to approximate $f(2, 0)$.
3. Find an equation for the plane though the origin and parallel to the plane $2x - y + 3z = 14$.
4. Find and classify all critical points of the function $f(x, y) = x^4 + y^4 - 4xy + 2$.
5. Let $R$ be the region between the curve $y = \sqrt{x}$ and the $x$-axis for $0 \leq x \leq 1$. Find $\iint_R \frac{2y}{x^2+1}\, dA$.

1. Find the second degree Taylor polynomial at the point $(1, 2)$ for the function $f(x, y) = x^2y + y$.
   **Solution:** $4 + 2(y - 2) + 4(x - 1) + \frac{1}{2}(4(x - 1)^2 + 4(x - 1)(y - 1))$
2. Use the second degree Taylor polynomial at the point $(1, 2)$ for the function $f(x, y) = x^2y + y$ to approximate $f(2, 0)$.
   **Solution:** $f(2, 0) \approx T_2(2, 0) = 2$
3. Find an equation for the plane though the origin and parallel to the plane $2x - y + 3z = 14$.
   **Solution:** $2x - y + 3z = 0$
4. Find and classify all critical points of the function $f(x, y) = x^4 + y^4 - 4xy + 2$.
   **Solution:** Saddle at $(0, 0)$, min at $(1, 1)$, min at $(-1, -1)$.
5. Let $R$ be the region between the curve $y = \sqrt{x}$ and the $x$-axis for $0 \leq x \leq 1$. Find $\iint_R \frac{2y}{x^2+1}\, dA$.
   **Solution:** $(\ln 2)/2$

6. Find the volume under the cone $z = \sqrt{x^2 + y^2}$ and above the disk $x^2 + y^2 \leq 4$.
7. Compute $\iiint_R x^2 + y^2 + z^2 \, dV$ where $R$ is the unit ball $x^2 + y^2 + z^2 \leq 1$.
8. Let $r(t) = (t^3, t, t^2)$, $0 \leq t \leq 1$. Compute $\int_r x^2 y \sqrt{z} \, dz$.
9. Let $C$ be a curve from $(1, 0, -2)$ to $(4, 6, 3)$. Let $f(x, y, z) = xyz + z^2$. Compute $\int_C \nabla f \cdot dr$.
10. Let $C$ be the sides of the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, $(0, 1)$ oriented counter-clockwise. Find $\int_C e^y \, dx + 2xe^y \, dy$.
11. Let $f(x, y, z) = x^2 + 2y^3 z$. Find the direction $f$ is decreasing most rapidly at the point $(0, 1, 0)$

6. Find the volume under the cone $z = \sqrt{x^2 + y^2}$ and above the disk $x^2 + y^2 \leq 4$.
   **Solution:** $16\pi/3$
7. Compute $\iiint_R x^2 + y^2 + z^2 \, dV$ where $R$ is the unit ball $x^2 + y^2 + z^2 \leq 1$.
   **Solution:** $4\pi/5$
8. Let $r(t) = (t^3, t, t^2)$, $0 \leq t \leq 1$. Compute $\int_r x^2 y \sqrt{z} \, dz$.
   **Solution:** $1/5$
9. Let $C$ be a curve from $(1, 0, -2)$ to $(4, 6, 3)$. Let $f(x, y, z) = xyz + z^2$. Compute $\int_C \nabla f \cdot dr$.
   **Solution:** 77
10. Let $C$ be the sides of the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, $(0, 1)$ oriented counter-clockwise. Find $\int_C e^y \, dx + 2xe^y \, dy$.
    **Solution:** $e - 1$
11. Let $f(x, y, z) = x^2 + 2y^3 z$. Find the direction $f$ is decreasing most rapidly at the point $(0, 1, 0)$
    **Solution:** $(0, 0, -1)$

12. Find where the tangent plane to the surface $x^2 + y^3 + z^4 = 10$ at the point $(3, 0, 1)$ hits the $z$-axis.
13. Find $\iint_R x \, dA$ where $R$ is the interior fo the triangle with vertices $(0, 0)$, $(0, 2)$ and $(1, 2)$.
14. Find the volume of the solid above the plane $z = 1$, below the surface $z = 2 + x^2 + y^2$ and enclosed by $x^2 + y^2 = 1$.
15. Find the projection of $(3, 1, 7)$ onto $(-2, 2, 1)$.
16. Find the volume under the graph of $f(x, y) = xy$ and above the triangle in the $xy$-plane with vertices $(0, 1, 0)$, $(2, 1, 0)$ and $(2, 2, 0)$.
17. Find the average value of $f(x, y, z) = x^2$ over the region bounded by a sphere of radius 1 around the origin.

12. Find where the tangent plane to the surface $x^2 + y^3 + z^4 = 10$ at the point $(3, 0, 1)$ hits the $z$-axis.
    **Solution:** $(0, 0, 11/2)$
13. Find $\iint_R x \, dA$ where $R$ is the interior fo the triangle with vertices $(0, 0)$, $(0, 2)$ and $(1, 2)$.
    **Solution:** $1/3$
14. Find the volume of the solid above the plane $z = 1$, below the surface $z = 2 + x^2 + y^2$ and enclosed by $x^2 + y^2 = 1$.
    **Solution:** $3\pi/2$
15. Find the projection of $(3, 1, 7)$ onto $(-2, 2, 1)$.
    **Solution:** $\frac{1}{3}(-2, 2, 1)$
16. Find the volume under the graph of $f(x, y) = xy$ and above the triangle in the $xy$-plane with vertices $(0, 1, 0)$, $(2, 1, 0)$ and $(2, 2, 0)$.
    **Solution:** $11/6$
17. Find the average value of $f(x, y, z) = x^2$ over the region bounded by a sphere of radius 1 around the origin.
    **Solution:** $1/5$

18. True/False

(a) Suppose $C$ is the positively oriented boundary of the region $R$. Then the area $A = \int_C -y \, dx$.

(b) Suppose $P(x, y)$ and $Q(x, y)$ and functions defined in a region and, in that region we have $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. Then, for any closed loop $C$ we have $\int_C P \, dx + Q \, dy = 0$.

(c) Suppose $f(x, y, z)$ has a local maximum at the point $(x_0, y_0, z_0)$ in the interior of its domain. Then, $\nabla f(x_0, y_0, z_0) = 0$.

(d) Suppose $(x_0, y_0, z_0)$ is in the interior of the domain of $f$ and $\nabla f(x_0, y_0, z_0) = 0$. Then $f$ has either a maximum or minimum at the point $(x_0, y_0, z_0)$.

(e) Suppose $f(x, y)$ and $g(x, y)$ both have the same domain and $\nabla f = \nabla g$ at all points in the domain. Then $f(x, y) = g(x, y)$ for all points in the domain.

(f) Suppose $f(x, y)$ and $g(x, y)$ both have the same domain and $\nabla f = \nabla g$ at all points in the domain. Then there is some constant $C$ so that $f(x, y) = g(x, y) + C$ for all points in the domain.

18. True/False

(a) Suppose $C$ is the positively oriented boundary of the region $R$. Then the area $A = \int_C -y\,dx$. **Solution:** True

(b) Suppose $P(x, y)$ and $Q(x, y)$ and functions defined in a region and, in that region we have $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. Then, for any closed loop $C$ we have $\int_C P\,dx + Q\,dy = 0$. **Solution:** False (need $R$ to be a rectangle.

(c) Suppose $f(x, y, z)$ has a local maximum at the point $(x_0, y_0, z_0)$ in the interior of its domain. Then, $\nabla f(x_0, y_0, z_0) = 0$. **Solution:** True

(d) Suppose $(x_0, y_0, z_0)$ is in the interior of the domain of $f$ and $\nabla f(x_0, y_0, z_0) = 0$. Then $f$ has either a maximum or minimum at the point $(x_0, y_0, z_0)$. **Solution:** False

(e) Suppose $f(x, y)$ and $g(x, y)$ both have the same domain and $\nabla f = \nabla g$ at all points in the domain. Then $f(x, y) = g(x, y)$ for all points in the domain. **Solution:** False

(f) Suppose $f(x, y)$ and $g(x, y)$ both have the same domain and $\nabla f = \nabla g$ at all points in the domain. Then there is some constant $C$ so that $f(x, y) = g(x, y) + C$ for all points in the domain. **Solution:** False

19. Find $\int_C x^2 z \, ds$ where $C$ is the line segment from $(0, 1, -1)$ to $(-2, 3, 0)$.
20. Let $C$ be the closed curve consisting of the line segment from $(-1, 0)$ to $(1, 0)$ and then the upper half of the unit circle back to $(-1, 0)$. Find $\int_C y^2 \cos x \, dx + (2xy + 2y \sin x) \, dy$.
21. Find the curl of $F = (z^2, zy, x^2 - z)$.
22. Find the center and radius of the sphere $x^2 + y^2 + z^2 - 6x + 2y + 6 = 0$
23. Find the arc length of the curve $r(t) = (2t\sqrt{t}, \cos 3t, \sin 3t), 0 \leq t \leq 3$.
24. Compute $\int_0^4 \int_{\sqrt{y}}^2 \sqrt{1 + x^3} \, dx \, dy$.

19. Find $\int_C x^2 z\, ds$ where $C$ is the line segment from $(0,1,-1)$ to $(-2,3,0)$.
**Solution:** $-1$

20. Let $C$ be the closed curve consisting of the line segment from $(-1,0)$ to $(1,0)$ and then the upper half of the unit circle back to $(-1,0)$. Find $\int_C y^2 \cos x\, dx + (2xy + 2y\sin x)\, dy$.
**Solution:** $4/3$

21. Find the curl of $F = (z^2, zy, x^2 - z)$.
**Solution:** $(0, -y, 2z - 2x)$

22. Find the center and radius of the sphere $x^2 + y^2 + z^2 - 6x + 2y + 6 = 0$
**Solution:** $(3, -1, 0)$, Radius 2.

23. Find the arc length of the curve $r(t) = (2t\sqrt{t}, \cos 3t, \sin 3t), 0 \le t \le 3$.
**Solution:** 14

24. Compute $\int_0^4 \int_{\sqrt{y}}^2 \sqrt{1 + x^3}\, dx\, dy$.
**Solution:** $\frac{52}{9}$

25. True/False
    (a) The vectors $(1, 3, -4)$ and $(1, 3, 4)$ are orthogonal.
    (b) The vector $(1, 0, 5)$ is longer than the vector $(3, -3, 1)$
    (c) $(1, 0, 1) \times (2, 3, 3)$ is parallel to $(3, 1, -3)$
26. Let $r(t) = (t^3, t, t^4)$, $0 \leq t \leq 1$. Compute $\int_C (3x + 8yz)\, ds$.
27. Let $R$ be the rectangle $[2, 4] \times [0, 1]$. Find a constant $M$ so that $\iint_R xe^y\, dA = M \cdot \mathrm{Area}(R)$.
28. Find the volume of the solid bounded by $x^2 + y^2 = 6$ and the planes $y + z = 7$ and $y + z = 14$.
29. Find and analyze all critical points of $f(x, y) = 5x^2y + \frac{5}{3}y^3 - 5x^2 - 5y^2 + 7$.

25. True/False

    (a) The vectors $(1, 3, -4)$ and $(1, 3, 4)$ are orthogonal. **Solution:** False

    (b) The vector $(1, 0, 5)$ is longer than the vector $(3, -3, 1)$ **Solution:** True

    (c) $(1, 0, 1) \times (2, 3, 3)$ is parallel to $(3, 1, -3)$ **Solution:** True

26. Let $r(t) = (t^3, t, t^4)$, $0 \leq t \leq 1$. Compute $\int_C (3x + 8yz)\, ds$.
**Solution:** $\frac{1}{18}(26\sqrt{26} - 1)$

27. Let $R$ be the rectangle $[2, 4] \times [0, 1]$. Find a constant $M$ so that $\iint_R xe^y\, dA = M \cdot \mathrm{Area}(R)$.
**Solution:** $3(e - 1)$

28. Find the volume of the solid bounded by $x^2 + y^2 = 6$ and the planes $y + z = 7$ and $y + z = 14$.
**Solution:** $42\pi$

29. Find and analyze all critical points of $f(x, y) = 5x^2y + \frac{5}{3}y^3 - 5x^2 - 5y^2 + 7$.
**Solution:** Max at $(0, 0)$, min at $(0, 2)$, saddles at $(1, 1)$ and $(-1, 1)$.