Math 233 - December 4, 2009
Solutions

- Center of Mass and Moments

$$
\begin{aligned}
M &= \iint_R \delta(x,y)\,dA \\
M_y &= \iint_R x\,\delta(x,y)\,dA \\
M_x &= \iint_R y\,\delta(x,y)\,dA \\
\overline{x} &= \frac{M_y}{M} \\
\overline{y} &= \frac{M_x}{M} \\
I_z &= \iint_R r^2\delta(x,y)\,dA
\end{aligned}
$$

1. Let R be the "ice cream cone" inside the sphere $\rho = 1$ and inside the cone $\phi = \pi/6$ (with $z \geq 0$). Suppose that $\delta(x,y,z) = z$. Compute the integrals:

$$
\begin{aligned}
M &= \iiint_R \delta(x,y,z)\,dV = \frac{\pi}{16} \\
M_{yz} &= \iiint_R x\delta(x,y,z)\,dV = 0 \\
M_{xz} &= \iiint_R y\delta(x,y,z)\,dV = 0 \\
M_{yz} &= \iiint_R z\delta(x,y,z)\,dV = \frac{\pi}{60}(8-3\sqrt{3})
\end{aligned}
$$

Lecture Problems

2. Find the center of mass of the tetrahedron with vertices $(0,0,0)$, $(3,0,0)$, $(0,3,0)$ and $(0,0,3)$ and with density $\delta(x,y,z) = x+y+z+1$.

   Note that $M_{yz} = M_{xz} = M_{xy}$.

$$
\begin{aligned}
M &= \int_0^3 \int_0^{3-x} \int_0^{3-x-y} (x+y+z+1)\,dz\,dy\,dx = \frac{117}{8} \\
M_{xy} &= \int_0^3 \int_0^{3-x} \int_0^{3-x-y} z(x+y+z+1)\,dz\,dy\,dx = \frac{459}{40} \\
\overline{x} &= \overline{y} = \overline{z} = \frac{51}{65}
\end{aligned}
$$

3. Let R be the region bounded by $x^2+y^2 = z^2$ and $x^2+y^2 = 4y$. Set up the integral using spherical coordinates.

$$\iiint_R \frac{1}{x^2+y^2+z^2}\, dV =$$

$$\int_{\pi/4}^{3\pi/4} \int_0^{\pi} \int_0^{4\sin\theta/\sin\phi} \sin\phi \; d\rho \; d\theta \; d\phi = 4\pi$$